ON THE MODULUS OF THE DERIVATIVE OF A POLYNOMIAL*

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1. *Introduction.* Let $P_n(z)$ be an arbitrary polynomial of degree n in z and let $|P_n(z)| \leqq M$ on a set C. The modulus of $P_n'(z)$† on C has an upper bound depending on M, on n, and on the set C. In this connection A. Markoff‡ has proved the following theorem.

Let $|P_n(z)| \leqq 1$ *in the interval* $-1 \leqq z \leqq +1$. *Then* $|P_n'(z)| \leqq n^2$ *for* $-1 \leqq z \leqq +1$. *This bound is attained only by the polynomial* $\pm \alpha \cos n \arccos z$, $|\alpha| = 1$.

A second fundamental result is the following theorem of S. Bernstein.§

Let $|P_n(z)| \leqq 1$ *on* C: $|z| \leqq 1$. *Then* $|P_n'(z)| \leqq n$ *on* C. *This bound is attained only by the polynomial* αz^n, $|\alpha| = 1$.

These theorems have been generalized in various directions by P. Montel,|| G. Szegö,¶ Dunham Jackson,** and the author.†† Here we will prove the following generalization.

THEOREM A. *Let* $P_n(z)$, *a polynomial of degree* n *in* z, *be in modulus less than a constant* M *on a set* C *which has no isolated points and whose complement has finite connectivity. Then*

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† $P_n'(z)$ denotes the first derivative of $P_n(z)$.

‡ A. Markoff, Abhandlungen der Akademie der Wissenschaften zu St. Petersburg, vol. 62 (1889), pp. 1–24. Markoff considers only polynomials with real coefficients. For the general case see M. Riesz, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368; see especially p. 357.

§ S. Bernstein, *Leçons sur les Propriétés Extrémales*, 1926, pp. 44–46.

|| P. Montel, Bulletin de la Société Mathématique de France, vol. 46 (1919), pp. 151–196.

¶ G. Szegö, Mathematische Zeitschrift, vol. 23 (1925), pp. 45–61.

** Dunham Jackson, this Bulletin, vol. 36 (1930), pp. 851–857; vol. 37 (1931), pp. 883–890.

†† W. E. Sewell, Proceedings National Academy of Sciences, vol. 21 (1935), pp. 255–258.

$|P_n'(z)| \leqq MK(C)n^2$ on C, where the constant $K(C)$ depends only on C.

An application* of this result gives the following theorem.

THEOREM B. *Let $f(z)$ be a function of z defined on C. If for every n there exists a polynomial of degree n, such that*

$$|f(z) - P_n(z)| < \frac{M}{n^\alpha}, \qquad (\alpha > 2),$$

on C, α and M independent of n and z, then $f(z)$ has a first derivative on C.

2. *Proof of Theorem A*. The result is obvious if $P_n(z)$ is a constant and in all other cases the set C is bounded. The theorem is not true for an isolated point.† Let C^ν be any one of the finite number of pieces composing the set C; the complement D^ν of C^ν is simply connected and contains the point $z=\infty$. Let $w=\phi(z)$ map D^ν conformally on $|w|>1$ so that the points at infinity correspond. Call the image of $|w|=1+\rho$, $(\rho>0)$, under the inverse map, C_ρ^ν. By a result of the author‡ it is known that the distance from C^ν to C_ρ^ν is at least as great as $B(C^\nu)\rho^2$, where $B(C^\nu)$ is a constant depending only on C^ν. Also by Walsh's§ generalization of a theorem of Bernstein, it is true that $|P_n(z)| \leqq M(1+\rho)^n$, z on or within C_ρ^ν. Now let z_0 be any point on C^ν and describe a circle γ about z_0 of radius $B(C^\nu)\rho^2$. Then

$$P_n'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{P_n(t)dt}{(t-z_0)^2}.$$

Since γ lies interior to C_ρ^ν, we know that $|P_n(t)| \leqq M(1+\rho)^n$, for t on γ, and consequently

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* Similar results for generalized derivatives in the case of functions of a real variable have been established by P. Montel, loc. cit. Montel's results have been extended to the complex domain for various types of regions by the author, loc. cit. For applications of Bernstein's and Markoff's theorems to approximation in the sense of least mth powers see Dunham Jackson, loc. cit.

† Consider a polynomial at the origin, for example.

‡ Loc. cit., p. 257. Here the result is stated without proof.

§ J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Colloquium Publications of this Society, vol. 20, 1935, pp. 77–78.

$$\left| P_n'(z_0) \right| \leqq \frac{1}{2\pi} \frac{M(1+\rho)^n 2\pi B(C^\nu)\rho^2}{[B(C^\nu)\rho^2]^2} = \frac{M}{B(C^\nu)} \frac{(1+\rho)^n}{\rho^2}.$$

If we take* $\rho = 2/(n-2)$, $(n>2)$, we have

$$\left| P_n'(z_0) \right| \leqq MK(C^\nu)n^2, \qquad (n > 2),$$

where $K(C^\nu)$ is independent of n for $n>2$. Since we may consider a polynomial of degree $n-1$ as a polynomial of degree n with leading coefficient 0, the constant can be adjusted so that the inequality holds for all n.† There are only a finite number of pieces C^ν, and the proof is complete.

3. *Proof of Theorem B*. By hypothesis

$$\left| f(z) - P_n(z) \right| < \frac{M}{n^\alpha} \qquad \text{on } C,$$

$$\left| f(z) - P_{n+1}(z) \right| < \frac{M}{(n+1)^\alpha} \text{ on } C;$$

then

$$\left| P_{n+1}(z) - P_n(z) \right| < \frac{2M}{n^\alpha} \qquad \text{on } C.$$

Hence by Theorem A

$$\left| P_{n+1}'(z) - P_n'(z) \right| < \frac{2MK(C)(n+1)^2}{n^\alpha} \text{ on } C.$$

By taking $2^{m-1} \leqq n < 2^m$ and considering the remainder of the series‡

$$[P_{2^m}'(z) - P_n'(z)] + [P_{2^{m+1}}'(z) - P_{2^m}'(z)] + \cdots,$$

we can establish uniform convergence of the series

$$P_0'(z) + [P_1'(z) - P_0'(z)] + [P_2'(z) - P_1'(z)] + \cdots$$

for any $\alpha > 2$, and thus the proof is complete.

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* For the details of this method see P. Montel, loc. cit., and G. Szegö, loc. cit.

† J. L. Walsh suggested this method for including all n.

‡ For the details of this method see P. Montel, loc. cit.