WALSH ON APPROXIMATIONS

Interpolation and Approximation by Rational Functions in the Complex Domain. By J. L. Walsh. American Mathematical Society Colloquium Publications, vol. 20. New York, 1935. ix+381 pp.

In 1885, C. Runge, the outstanding exponent of applied mathematics in Germany, published a theorem concerning analytic functions which became the starting point of a long and fruitful evolution. It states, as does a classical theorem of Weierstrass about continuous functions of a real variable, the existence of polynomials approximating with a preassigned accuracy a given analytic function, regular in the closed interior of a curve. Runge's proof, although very elementary and ingenious, does not give a definite principle (see below) by which the approximating polynomials can be calculated, or, to use an equivalent formulation, by which the development of the given function in a polynomial series can be obtained. Later on various other proofs have been given, characterizing the polynomials in question by means of important additional properties. Among others, developments of the form

$$c_0p_0(z)+c_1p_1(z)+c_2p_2(z)+\cdots+c_np_n(z)+\cdots$$

were investigated, in which the polynomials $\{p_n(z)\}$ depend only on the curve considered and the coefficients $\{c_n\}$ on the individual function developed. An account of the older results in this direction was given by P. Montel in his book, *Leçons sur les Séries de Polynomes à une Variable Complexe* (Borel's Collection, 1910). Since that time, however, a further extremely intensive and in some respects very fascinating cultivation of this field can be observed. A short survey of the modern situation was given by Walsh in a small volume in the French series *Mémorial des Sciences Mathématiques* (vol. 73, 1935). The present book of Walsh, which appeared as the latest volume of the Colloquium Lectures, takes up the recent development on a wide basis. Walsh himself is participating in the advancement of this field to a considerable extent, partly by various investigations of his own, partly by his suggestions to other writers. He gives in this book not only a careful and detailed picture of the present stage but furnishes also some new contributions to the subject.

Generally speaking this book deals primarily with infinite sequences or series of polynomials and rational functions converging towards a given analytic function and fulfilling some additional conditions. These conditions are characterized either by the coincidence of the polynomial or rational function with the given analytic function in proper points (problem of interpolation) or lying as "close" to it as possible (problem of the best approximation). Moreover the connection of those interpolatory and approximative properties is investigated. The theory of interpolation and the theory of approximations are, of course, tremendously broad fields. Among others they have a branch in the pure real region taking up the properties of sequences of polynomials associated with a continuous function of a real variable possessing a given "degree" of continuity (S. Bernstein, de la Vallée Poussin, Jackson (see vol. 11 of these Colloquium Lectures)). The present book has no direct connections with these ideas. It considers sequences belonging exclusively to analytic functions. Furthermore, speaking again roughly, it investigates infinite sequences of polynomials (rational functions) convergent with the rapidity of the geometric series or having bounds of the form $O(R^n)$, R a fixed positive number, $n \to \infty$.

In order to give a more exact idea of the material treated in the book, we refer briefly to some more or less central results of this field.

The first proof of Runge's theorem associating a definite type of polynomial development with an analytic function is that given by Hilbert (1897). He uses to this end the so-called Jacobi series, proceeding in terms of the powers of a fixed polynomial $p(z)$, say of kth degree, and having coefficients which are polynomials of a degree $\leqq k-1$. Here $p(z)$ depends only on the given curve, the coefficients, however, on the individual function developed. In the case of a circle $p(z)$ is linear, and we obtain the customary power series development. Apart from the elegance of this result, Hilbert's proof was destined to influence the later evolution primarily because of another reason. It displays for the first time the connection of the general problem with potential theory and thus virtually also with conformal representation. He obtains indeed the polynomial $p(z)$, starting from the classical equilibrium problem of the logarithmic potential, dealt with by Gauss, and approximating the so-called "natural mass distribution" (possessing a constant potential interior to the curve) by means of discrete masses.

Faber was the first (1903) to use particular polynomials connected with the conformal representation of the region exterior to the given curve onto the exterior of a circle conserving the point at infinity. These polynomials can be calculated easily from the map function. Faber proves then the possibility of developing a given analytic function in a series of the form above indicated, $p_n(z)$ being the polynomials in question.

Fejér considers (1918) polynomials coinciding with a given analytic function, regular interior to and on a curve, in "uniformly distributed" points of this curve. The last term is meant in the sense of the conformal representation described before; the points in question correspond to points uniformly distributed on the circle in the ordinary sense. By previous important results of Runge (1903) it was well known that subdivision of the given curve in *equal arcs* does not lead in general to a convergent interpolation procedure. Fejér's subarcs are, indeed, equal in the sense that they contribute *equal masses* to the equilibrium or natural distribution mentioned above. Incidentally, Kalmar showed that this condition, at least in the asymptotical sense, is also necessary to the convergence of the interpolation procedure.

Faber (1920) investigates polynomials of a given degree, differing in the sense of Tchebycheff as little as possible from the given function (Tchebycheff polynomials). These polynomials have a simple asymptotic relation to the map function. Their explicit determination is however not possible, except in special cases.

Szegö introduces (1921) the corresponding polynomials in the sense of Bessel's approximation (orthogonal polynomials). They possess asymptotic properties similar to those of the Tchebycheff polynomials and can be calculated explicitly by means of Gram's orthogonalization procedure.

Fekete (1926) considers the product of all $C_{n,2}$ distances of n points, running independently of one another on the given curve. The system of n points giving a maximal value to this product, $n=2, 3, 4, \cdots$, can be used in the same way as in Fejér's case in place of the "uniformly distributed" points as interpolation points of a convergent procedure.

This short enumeration of special sequences and developments associated with a given analytic function illustrates the colorful variety of problems arising here. The reader finds in the book of Walsh not only an exhaustive treatment of these questions but also important refinements, generalizations, and analogs which are sometimes rather intricate and require great care.

Concerning the details the following indications may be sufficient. Chapters 1 and 2 start with various preparations from the theory of point sets and analytic functions. We find here the formulation of Runge's theorem and its extensions, Lindelöf's map function theorems, remarks concerning Cauchy's integral, approximations in general and related topics. Chapters 3 and 4 take up lemniscates, Jacobi series, and Green's functions, altogether the background of Hilbert's proof in the refined shape of modern analysis. In the subsequent Chapters 5 and 6 the best approximation of a given analytic function is treated, primarily in the sense of Tchebycheff and Bessel. The last problem is, of course, intimately connected with convergence in the mean. We find here a short treatment of the Fischer-Fr. Riesz theorem. Numerous generalizations of the Tchebycheff and orthogonal polynomials (in particular introducing a "norm" function on the curve) can be likewise defined. Chapter 7 is devoted to the problem of interpolation in "uniformly distributed" points. The questions of interpolation and approximation are revised in Chapters 8 and 9 replacing polynomials by rational functions with preassigned poles or with preassigned restrictions concerning the number and position of the poles. Lemniscates and Green's functions are here used again, in a more general form. The next chapter studies analytic functions regular in the unit circle and fulfilling there some interpolatory conditions. The problem is, roughly speaking, to find the function of this type having a maximum modulus (or square integral) as small as possible. A long list of investigations dealing with this and similar questions could be enumerated. The recent results of Nevanlinna and Denjoy are here presented. Chapter 11 takes up interpolation by polynomials under auxiliary conditions. The last chapter gives a survey of existence and uniqueness theorems in the case of Tchebycheff and other approximations.

The bibliography is fairly complete. The reviewer missed among others a reference to one of his own papers: *Bemerkungen*, $\cdots$ (Mathematische Zeitschrift, vol. 21 (1924), pp. 203–208). This contains not only a general proof of the coincidence of the transfinite diameter with the capacity but also the extension of the conception of uniformly distributed points to the case of a finite number of mutually exterior analytic curves.

The exposition of the book is clear, the form elegant, the style lucid and plain, the presentation careful and detailed, now and then almost too detailed. The reader may have the feeling that the author was sometimes hesitating as to whether the book should be destined for the beginner or whether its purpose should rather be a survey of a field for experts and fellow-contributors. A little contrast can be doubtless observed between the broad treatment of some comparatively elementary situations and the preference of special and individual topics to some others of importance. But at all events the book will be useful from both points of view and it is a significant enrichment of the series of the Colloquium Lectures.

GABRIEL SZEGÖ

MORSE ON CALCULUS OF VARIATIONS

The Calculus of Variations in the Large. By Marston Morse. Colloquium Publications of the American Mathematical Society, vol. 18. New York, 1934. ix+368 pp.

The background for the theory elaborated in this volume lies in two rather distinct fields of mathematics. We have on the one hand the theory of critical points of functions of n real variables, largely created and developed by the author and his students; on the other hand, the classical calculus of variations and its modern treatment as a part of the functional calculus, to which Hadamard and Tonelli have made the fundamental contributions. The origin of the theory of critical points is perhaps to be found in the minimax principle introduced by Birkhoff in his paper on *Dynamical systems with two degrees of freedom* (see Transactions of this Society, vol. 18 (1917), p. 240). While at this early stage the connection of the theory with the calculus of variations was made clear, later developments have made the relations between the two fields much more intimate.

The calculus of variations in the large derives its interest not only from its use of functional calculus methods. More significant is the fact that it undertakes a study of the configurations to which a calculus of variations problem gives rise, namely, the extremals, with respect to important properties apart from the question whether or not they furnish extreme values for a definite integral. In the classical calculus of variations, only such extremal arcs AB were considered which contained no point A' conjugate to A. The theory with which this book is concerned, and also the earlier investigations of Birkhoff, reveal the importance of extremal arcs AB upon which there may be one or more points conjugate to A; not only arcs in the small, restricted by the exclusion of conjugate points, but also arcs in the large have to be considered for a full understanding of this theory.

A critical point x_i^0 of $f(x_1, \cdots, x_m)$ is a point at which all the first order partial derivatives of f vanish. Each critical point has a type number equal to the number of negative terms in the quadratic form which represents the second order terms of $f(x_i)-f(x_i^0)$, when reduced to a sum of squares. The author's earlier papers establish a beautifully simple set of relations between the numbers of critical points of various types of $f(x_1, \cdots, x_m)$ and the connectivity numbers of the domain R over which the function ranges. Thus the topological character of the domain of the independent variable is linked up