A NECESSARY CONDITION FOR APPROXIMATION BY RATIONAL FUNCTIONS

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It is the object of the present note to establish the following two theorems; terminology is uniform with that of the writer's recent book on approximation:*

THEOREM 1. *In the extended z plane let R be a region whose boundary is denoted by B. Let every component of B either separate the plane into at least two regions or contain in each of its neighborhoods points of an infinite number of components of B each of which separates the plane into at least two regions. Let the function $f(z)$ be single-valued and analytic in R in the neighborhood of B, and let $\lim_{z_k \to z_0} f(z_k)$ exist and be equal to zero whenever the points z_k lie interior to R and approach a point z_0 of B. Then the function $f(z)$ vanishes identically interior to R in the neighborhood of B.*

THEOREM 2. *Let C be an arbitrary closed point set of the extended plane, and let points z_k (not necessarily denumerable) be given exterior to C. A necessary and sufficient condition that a function $f(z)$ single-valued and analytic on C can be uniformly approximated as closely as desired on C by a rational function whose poles lie in the points z_k is that $f(z)$ can be extended analytically from C so as to be single-valued and analytic in every point of the plane which is separated by C from the points z_k. That is to say, the condition is that there should exist a function which is single-valued and analytic not merely on C but also in every point of the plane separated by C from the points z_k, and which coincides with $f(z)$ on C.*

These theorems are slightly more general than the corresponding theorems that are given in the book just mentioned (loc. cit., §1.9, Theorem 15; §1.10, Theorem 16). The present Theorem 2 seems to be the definitive result in its field.

The sufficiency of the condition of Theorem 2 has already

* *Interpolation and Approximation by Rational Functions in the Complex Domain*, Colloquium Publications of this Society, vol. 20, 1935.

been established (loc. cit., §1.6, Theorem 8). The necessity of that condition will be proved by use of Theorem 1, and Theorem 1 is to be proved by application of quite recent results on harmonic measure due to R. Nevanlinna.*

I am indebted to my colleague Professor L. Ahlfors for this proof of Theorem 1. I formulated Theorem 1 as something more than a conjecture to Professor Ahlfors; he at once suggested the proof now to be given.

Assume B finite, which involves no loss of generality. The neighborhood of B in Theorem 1 may be chosen (loc. cit., §1.3, Theorem 4, and method of §1.5, Theorem 7) as a finite number of mutually exclusive subregions $S_1, S_2, \cdots, S_\mu$ of R, each region S_k bounded interior to R by a single Jordan curve, and bounded otherwise by one or more components of B. Every point z interior to R and within a certain distance $\delta>0$ of B lies interior to some S_k.

The capacity of a point set is monotonically non-decreasing with the point set. The boundary of any limited region has positive capacity. Each component of B either separates the plane into at least two regions or contains limit points of components of B so separating the plane. Hence the totality of components of B which bound each S_k have positive capacity. The function $f(z)$ is bounded in S_k, if the Jordan curves bounding all the regions S_j are suitably chosen. Consequently (Nevanlinna, loc. cit.; $f(z)$ is bounded in S_k, hence *beschränktartig*; its boundary values on B vanish, thus form a set of harmonic measure zero) the function $f(z)$ vanishes identically in each S_k, therefore vanishes identically throughout the given neighborhood of B.

Theorem 1 is now completely proved and will serve in the proof of the necessity of the condition of Theorem 2.

Let $r_n(z)$ be a sequence of rational functions whose poles lie in the prescribed points z_k and which converges uniformly to $f(z)$ on C. Let R' be any one of the regions into which C separates the plane which is also separated by C from the points z_k, and let B' be the boundary of R'. Let S be the point set consisting of the components of B' each of which effectively

* Proceedings of the Eighth Scandinavian Congress (Stockholm, 1934) of Mathematicians (Lund, 1935), pp. 116–133, especially pp. 129–130.

separates R' from points z_k, and let $\bar{S}$ be the set composed of S plus its limit points. Every point of $\bar{S}$ is a point of B', hence a point of C. Let R be the region bounded by $\bar{S}$, such that every point of R' is a point of R. Every component of $\bar{S}$ either separates the plane into at least two regions or contains in each of its neighborhoods points of an infinite number of components of $\bar{S}$ each of which separates the plane into at least two regions.

The sequence $r_n(z)$ converges uniformly to $f(z)$ on C and on $\bar{S}$. Each function $r_n(z)$ is analytic in the closed region $\bar{R}=R+\bar{S}$, so the sequence $r_n(z)$ converges uniformly in the closed region $\bar{R}$ to some function $F(z)$, which is analytic in R and continuous in $\bar{R}$, and coincides with $f(z)$ on $\bar{S}$. The function $f(z)$ is analytic on C, thus (loc. cit., §1.5, Theorem 7) can be extended analytically from C into R in the neighborhood of $\bar{S}$. The function $F(z)-f(z)$ is analytic in the neighborhood of $\bar{S}$ and approaches zero whenever z interior to R approaches $\bar{S}$. It follows from Theorem 1 that $F(z)-f(z)$ vanishes identically in the neighborhood of $\bar{S}$, so the analytic extension of $f(z)$ from $\bar{S}$ or from C into the interior of R coincides with $F(z)$. This is true for every region R, so the proof is complete. Indeed, we have shown that $f(z)$ can be extended from C so as to be single-valued and analytic not merely throughout every region R' but throughout every region R.

From Theorem 2 follows without difficulty (loc. cit., §1.10, Theorem 17) the following theorem.

THEOREM 3. *Let C be an arbitrary closed point set of the extended plane, and let points z_k (not necessarily denumerable) be given not belonging to C. A necessary and sufficient condition that every function $f(z)$ analytic on C can be uniformly approximated as closely as desired on C by a rational function whose poles lie in the points z_k is that at least one point z_k lie in each of the regions into which C separates the plane.*

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