LINEAR DIFFERENTIAL EQUATIONS OF INFINITE ORDER*

BY R. D. CARMICHAEL

1. *Introduction.* In this address we shall give such a conspectus of the theory of linear differential equations of infinite order as will enable the reader rapidly to orient himself with respect to this subject. Neither in the presentation of results already developed nor in the bibliographical references (in the footnotes) is there any attempt to attain an exhaustive account. The purpose is rather that of a general outlook on the subject such as may interest a considerable number of persons and may serve as a point of departure for a few investigators who may desire to penetrate a relatively new and unexplored domain, the importance of which will certainly be more fully recognized as the subject is further developed in the next two or three decades.

In §2 the general nature of the problem of linear differential equations of infinite order is indicated partly by means of examples and partly by means of notions of a general character which are intimately associated with the somewhat more special problem which is the center of our present interest. In §3 are set forth some of the many connections of this problem with other matters of more or less wide interest in the field of analysis. A brief account of the present state of knowledge with reference to linear differential equations of infinite order with constant coefficients is given in §4, while §§5 and 6 treat (but with somewhat less fullness) the corresponding matters for equations with polynomial and with analytic coefficients, respectively. Finally, §7 is devoted to a brief account of some further problems and connections of the theory of differential equations of infinite order.

2. *Nature of the Problem.* A typical problem in the theory of linear differential equations of infinite order is that of solving the equation

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots = \phi(x), \tag{1}$$

* Retiring address of the Chairman of Section A of the A.A.A.S., delivered at St. Louis, December 31, 1935.

where $\phi(x)$, $a_0(x)$, $a_1(x)$, $\cdots$ are given functions of x subject to suitable conditions and where y is the function to be determined, the primes denoting differentiation with respect to x. In the investigation of such equations the unknown function $y(x)$ is often required to belong to a particular class of functions such, for instance, as the functions of exponential type.

One may also investigate systems such as the following:

$$\frac{dy_i}{dx} = \sum_{j=1}^{\infty} A_{ij}(x)y_j(x) + \Phi_i(x), \qquad (i = 1, 2, 3, \cdots), \tag{2}$$

where the A's and the Φ's are given functions and the y's are the functions to be determined.

In the infinite case the analogy between (1) and (2) does not seem* to be as close as that between the corresponding equation and system in the finite case. Thus, if we consider the special system

$$\alpha_0(x)\frac{dy_1}{dx} + \alpha_1(x)y_1 + \alpha_2(x)y_2 + \cdots + \alpha_n(x)y_n + \cdots = 0,$$

$$\frac{dy_2}{dx} = y_1, \quad \frac{dy_3}{dx} = y_2, \quad \cdots, \quad \frac{dy_n}{dx} = y_{n-1}, \quad \cdots,$$

and if we write y for y_n, then the functional equation for y may be indicated by the following relation which involves a limiting process as to n:

$$\left[\alpha_0(x)y^{(n)}+\alpha_1(x)y^{(n-1)}+ \cdots +\alpha_{n-1}(x)y'+\alpha_n(x)y\right]_{n=\infty}=0. \tag{3}$$

There is a marked difference between the theories of (1) and (3).

Systems of equations of the form

$$\sum_{k=1}^{m}\sum_{\nu=0}^{\infty} a_{ik\nu}(x)y_k^{(\nu)} = \phi_i(x), \qquad (i = 1, 2, \cdots, m), \tag{4}$$

have been considered both when m is finite and when m is infinite. Systems (2) are capable of obvious generalizations.

In a few instances linear partial equations of infinite order have appeared. Non-linear ordinary equations of infinite order have also been investigated to some extent.

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* See T. Lalesco, Journal de Mathématiques, (6), vol. 4 (1908), pp. 125–202, especially pp. 182–183.

For the sake of unity we shall direct our attention mainly (but not entirely) to (1) and (4), particularly since by means of them we shall be able to set forth that part of the theory which up to the present has received the major share of attention.

Two pioneers in the development of the theory of linear differential equations of infinite order were S. Pincherle* and C. Bourlet,† the priority belonging to the former. But the work of Bourlet was independent of that of Pincherle. In fact, the two authors approached the problem from quite different points of view; but their results overlapped in important ways. Recently P. Flamant‡ has further investigated the problem from the point of view of Bourlet and Pincherle. Bourlet used the term *transmutation* to denote an operation T which makes a given function $\phi(x)$, the object of the operation, correspond to another function $T\phi(x)$, the result of the operation. The transmutation T is said to be distributive if for arbitrary functions $\phi(x)$ and $\psi(x)$ and for an arbitrary constant c, we have

$$T[\phi(x) + \psi(x)] = T\phi(x) + T\psi(x), \quad T[c\,\phi(x)] = cT\phi(x).$$

Pincherle simply calls T a distributive operation. Bourlet introduces at once a certain notion of continuity with respect to the transmutation, but his account of the matter is lacking in clarity. Both Bourlet and Pincherle insist on the proposition that every additive, uniform, continuous, and "regular" transmutation can be represented by a series of the form

$$Tu = \sum_{n=0}^{\infty} a_n(x) \frac{d^n u}{dx^n},$$

thus bringing the theory of transmutations into intimate association with the theory of differential equations of infinite order. Flamant (loc. cit.) has undertaken to subject transmutations to a more rigorous analysis than that which satisfied the earlier

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* For Pincherle's contributions to this problem see the outline of his work in Acta Mathematica, vol. 46 (1925), pp. 341–362.

† For Bourlet's work, see especially Annales de l'École Normale Supérieur, (3), vol. 14 (1897), pp. 133–190; (3), vol. 16 (1899), pp. 333–375; and the papers of Flamant mentioned in the next note.

‡ For papers by Flamant, see especially Palermo Rendiconti, vol. 48 (1924), pp. 135–208; Bulletin des Sciences Mathématiques, (2), vol. 52 (1928), pp. 26–48, 77–96, 104–128; Palermo Rendiconti, vol. 54 (1930), pp. 371–413.

investigators and in particular to clarify the situation as regards the proposition just quoted by giving precise and sufficiently general conditions for its validity. He also generalizes the formula by replacing the operation D of differentiation by certain more general operations L.

The particular transmutation Tu,

$$Tu = u - \frac{x}{1!}u' + \frac{x^2}{2!}u'' + \cdots + (-1)^n \frac{x^n}{n!}u^{(n)} + \cdots ,$$

will bring to notice (see Bourlet, loc. cit., 1897, pp. 174–176) one aspect of the difference between equations of finite order and those of infinite order. For functions analytic at $x=0$ one has $Tu=u(0)$. Hence this transmutation makes every such function correspond to a constant. The inversion of the transmutation is therefore generally impossible in the domain of functions analytic at $x=0$; it is only constants which admit inverses. If a denotes an arbitrary constant, then the equation $Tu=a$ is a linear differential equation of infinite order which is verified by the function $u=a+xf(x)$, where $f(x)$ is an *arbitrary* function analytic at $x=0$.

A much more instructive example is the following one:*

$$(1-x)y + y' + \cdots + \frac{y^{(n)}}{n!} + \cdots = 0.$$

The classic function $\Gamma(x)$ satisfies this equation in every region of the plane exterior to the circles of radius 1 about the points $0, -1, -2, \cdots, -n, \cdots$ as centers, since for such regions the equation (for suitable functions) reduces to the difference equation $y(x+1)-xy(x)=0$. But, in a point situated in the interior of one of the preceding circles, the development $\sum y^{(n)}(x)/n!$ is evidently divergent for $y=\Gamma(x)$, although the product $x\Gamma(x)$ is everywhere finite in one of these circles save at the center. Therefore the given differential equation is not satisfied by $y=\Gamma(x)$ in the interior of one of these circles. This circumstance is new and is characteristic of linear differential equations of infinite order. It is therefore not sufficient to be assured that a non-integral analytic function satisfies a linear differential equation of infinite order in a point in order to conclude

* See Lalesco, loc. cit., 1908, pp. 194–195.

that it satisfies the equation throughout the domain of its existence. Lalesco (loc. cit.) investigates the problem which is set by the existence of such situations as this example brings to light. The reader is referred to his memoir for the results.

3. *Connections of the Problem.* Pincherle,* in one of his early papers, has indicated certain interesting connections of the problem of linear differential equations of infinite order. Let $A(z)$ denote the function

$$A(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}},$$

which is regular outside of the circle of radius R about 0 as center. Put $z = y - x$. Then the function $A(y-x)$ is regular when $|y-x| > R$, and hence when either $|y| > |x| + R$ or $|x| > |y| + R$. Under the first of the last two hypotheses, if we take x interior to a circle of radius σ about 0 as center and y exterior to the circle of radius $R+\sigma$ about 0 as center, then we may write $A(y-x)$ in either of the forms

$$A(y - x) = \sum_{n=0}^{\infty} \frac{a_n}{(y - x)^{n+1}}, \tag{5}$$

$$A(y - x) = \sum_{n=0}^{\infty} \frac{A_n(x)}{y^{n+1}}, \tag{6}$$

where the $A_n(x)$ constitute the system of Appell polynomials with the coefficients a_n (and satisfying the relation $A_n'(x) = nA_{n-1}(x)$). Under the second of the hypotheses, if we take y interior to the named circle of radius σ, and x exterior to the named circle of radius $R+\sigma$, then we have for $A(y-x)$ either the development (5) or the development

$$A(y - x) = \sum_{n=0}^{\infty} \frac{y^n}{n!} A^{(n)}(- x), \tag{7}$$

where $A^{(n)}(z)$ is the nth derivative of $A(z)$.

One now considers the expression

* Pincherle, Memorie Reale Accademia delle Scienze, Bologna, (4), vol. 9 (1888), pp. 45–71; French translation in Acta Mathematica, vol. 48 (1926) pp. 279–304.

$$A(\psi) = \int_{(l)} A(y - x)\psi(y)dy, \tag{8}$$

where the integration is taken along a line (l) of the y plane, and $\psi(y)$ is an analytic function without singularities along (l). One distinguishes two cases as follows:

(a) The line (l) (either finite in length or infinite under suitable conditions) is such that the modulus of each point on it exceeds $R+\sigma$. Then, for $|x| <\sigma$, one can admit for $A(y-x)$ either of the developments (5) and (6). On putting

$$\int_{(l)} \frac{\psi(y)dy}{y - x} = \phi(x), \tag{9}$$

one has from (8), in the respective cases, the developments

$$A(\psi) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \phi^{(n)}(x), \tag{10}$$

$$A(\psi) = \sum_{n=0}^{\infty} c_n A_n(x), \qquad c_n = \int_{(l)} \frac{\psi(y)dy}{y^{n+1}}. \tag{11}$$

(b) The line (l) is such that all its points are interior to the circle of radius σ about 0 as center. Suppose that $|x| >R+\sigma$. For $A(y-x)$ we may then use either of the developments (5) and (7). Then from (9) we have, in the respective cases, the relations

$$A(\psi) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \phi^{(n)}(x), \tag{12}$$

$$A(\psi) = \sum_{n=0}^{\infty} c_n' \frac{A^{(n)}(- x)}{n!}, \qquad c_n' = \int_{(l)} y^n \psi(y)dy. \tag{13}$$

Now let $f(x)$ be a given function and consider the functional equation

$$A(\psi) = f(x). \tag{14}$$

The solution of this equation involves the inversion of the integral in (8). The formulas which have been given show that this problem, at least in many of its essential elements, coincides with other functional problems as follows:

(a) to solve a linear differential equation of infinite order with constant coefficients, namely, the equation

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} \phi^{(n)}(x) = f(x);$$

(b) to find the development of a given function of x in terms of a given system of Appell polynomials;

(c) to develop a given function $f(x)$ in a series in terms of the successive derivatives of a given function $A(-x)$.

In the paper here cited Pincherle makes an important contribution toward the solution of these problems. He has also treated them and other closely related matters in many other memoirs whose dates of publication now stretch over almost the whole of a half-century period.

H. T. Davis* has given a method for deriving the Fredholm theory from the theory of differential equations of infinite order.

I. M. Sheffer† and others have indicated the close connection which exists between the theory of a system of infinitely many linear equations in infinitely many unknowns and the theory of linear differential equations of infinite order. To pass from the latter to the former, when working in the field of analytic functions, one has only to assume a power series expansion for the unknown function, substitute into the differential equation, and equate coefficients. Conversely, if one is given the system of equations

$$\sum_{j=0}^{\infty} a_{ij} y_j = c_i, \qquad (i = 0, 1, \cdots),$$

where the a_{ij} and the c_i are wholly arbitrary, then there exists a linear differential equation

$$P_0(x)y(x) + P_1(x)y'(x) + \cdots + P_n(x)y^{(n)}(x) + \cdots = C(x),$$

where the expressions

* H. T. Davis, Annals of Mathematics, vol. 28 (1927), pp. 309–317.

† I. M. Sheffer, Annals of Mathematics, vol. 30 (1929), pp. 250–264. See also, among others, E. Hilb, Mathematische Annalen, vol. 82 (1920), pp. 1–39, vol. 84 (1921), pp. 16–30, 43–52, vol. 85 (1922), pp. 88–89; O. Perron, Mathematische Annalen, vol. 84 (1921), pp. 1–15, 31–42; F. Lettenmeyer, Dissertation, München, 1927, pp. 1–52.

$$C(x) = \sum_{k=0}^{\infty} c_k x^k \qquad \text{and} \qquad P_i(x) = \sum_{k=0}^{\infty} p_{ik} x^k$$

are formal power series in x such that on setting $y(x) = \sum y_k x^k$ and substituting into the differential equation and equating coefficients the resulting system is identical with the given one; and it is easy to obtain the relations between the p's and the a's to effect this formal equivalence.

D. C. Lewis* developed the theory of infinite systems of ordinary non-linear differential systems of infinite order with applications to certain second order partial differential equations, using for this purpose certain integral forms which are equivalent to the given systems with specified initial conditions.

In a paper on the Laplace differential equation of infinite order, namely, the equation

$$\sum_{n=0}^{\infty} (a_{n0} + a_{n1}x + a_{n2}x^2 + \cdots + a_{np}x^p)u^{(n)}(x) = f(x),$$

where p is a positive integer and not all the quantities a_{np} are zero, H. T. Davis† pointed out that the general theory of this equation formally unifies the theories of the following types of functional equations where the $p_i(x)$ are polynomials of degree not greater than p:

(a) $$u(x) + \int_x^{\infty} \sum_{i=1}^{m} p_i(x)\phi_i(t - x)u(t)dt = f(x),$$

where the $\phi_i(x)$ are such that the integrals $\int_x^{\infty}\phi_i(s)s^n ds$ exist for all values of n;

(b) $$u(x) + \int_a^b \sum_{i=1}^{m} p_i(x)q_i(t)u(x + ct)dt = f(x);$$

(c) $$p_m(x)u(x + m) + p_{m-1}(x)u(x + m - 1) + \cdots + p_0(x)u(x) = f(x);$$

(d) the Laplace differential equation of finite order.

These formal equivalences are put in evidence by means of expansions in Taylor's series.

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* D. C. Lewis, Transactions of this Society, vol. 35 (1933), pp. 792–823.

† H. T. Davis, Annals of Mathematics, vol. 32 (1931), pp. 686–714.

4. *Equations with Constant Coefficients.* Here the first problem is that of solving the equation

$$(15) \qquad a_0 y + a_1 y' + a_2 y'' + \cdots = \phi(x),$$

where the coefficients a_ν are constants, $\phi(x)$ is a given function and $y(x)$ is the function to be determined. If $y(x)$ is any solution, then the left member of equation (15) must be a convergent series; hence we must have

$$(16) \qquad \limsup_{\nu=\infty} \left| a_\nu y^{(\nu)}(x) \right|^{1/\nu}$$

less than or equal to unity for every admissible value of x. When this superior limit is 1 there is uncertainty about convergence. Hence in a first study of (15) it seems natural to require that the superior limit (16) shall have a value less than 1; and we now impose this condition.

It seems natural also, in a first view of the problem, to attain this condition by hypotheses on the a_ν and the $y(x)$ taken separately, since the former are known constants and the latter is the unknown function. We may naturally bring this about by restricting the solutions $y(x)$, which are to be admitted, to functions $y(x)$ such that the superior limit

$$(17) \qquad \limsup_{\nu=\infty} \left| y^{(\nu)}(x) \right|^{1/\nu}$$

shall be finite. Then we are led naturally to the following condition on the coefficients:

$$(18) \qquad \limsup_{\nu=\infty} \left| a_\nu \right|^{1/\nu} = \sigma < \infty .$$

We therefore adopt this condition as one of the basic hypotheses.

The class of functions $y(x)$ and the basic hypothesis on the coefficients a_ν, to which we have thus been led in a natural way, have as a matter of fact played a central role in the theory of equations of the form (1).

If $y(x)$ is an analytic function, it is readily proved that the value of the superior limit (17) is independent of x. If this value is τ, we say that $y(x)$ is of exponential type τ.

The most elegant known theorem concerning solutions of equation (15) is perhaps the following.

THEOREM 1. *In the linear differential equation of infinite order,*

$$a_0 y + a_1 y' + a_2 y'' + \cdots = \phi(x), \tag{15}$$

let the coefficients a_ν *be constants such that the function* $F(z)$,

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \tag{19}$$

is analytic in the region $|z| \leqq q$, *where* q *is a given positive constant or zero, and let* $\phi(x)$ *be a function of exponential type not exceeding* q. *If* $F(z)$ *vanishes at least once in the region* $|z| \leqq q$, *let* n *be the number of its zeros in this region (each counted according to its multiplicity) and let* $P(z)$ *be the polynomial of degree* n *with leading coefficient unity such that* $F(z)/P(z)$ *does not vanish in the region. If* $F(z)$ *does not vanish in the region, let* $P(z)$ *be identically equal to* 1. *When* $P(z) \equiv 1$, *let* $P_{n-1}(z)$ *be identically equal to zero; otherwise let it be an arbitrary polynomial of degree* $n-1$ *(including the case of an arbitrary constant when* $n=1$). *Then the general solution* $y(x)$ *of* (15), *subject to the condition that it shall be a function of exponential type not exceeding* q, *may be written in the form*

$$y(x) = \frac{1}{2\pi i}\int_{C_\rho} \frac{\psi(s)}{F(s)} e^{xs} ds + \frac{1}{2\pi i}\int_{C_\rho} \frac{P_{n-1}(s)}{P(s)} e^{xs} ds, \tag{20}$$

where

$$\psi(s) = \sum_{\nu=0}^{\infty} \frac{\phi^{(\nu)}(0)}{s^{\nu+1}}, \tag{21}$$

and where C_ρ *is a circle of radius* ρ *about* 0 *as center,* ρ *being greater than* q *and such that* $F(z)$ *is analytic in the region* $q > |z| \leqq \rho$ *and does not vanish there.*

If $\phi(x)$ *is precisely of exponential type* q, *then the named solution* $y(x)$ *is also of exponential type* q.

If we take $P_{n-1}(x) \equiv 0$, we have in (20) a particular solution of equation (15). On subtracting this particular solution from the general solution (20) we have the general solution (of exponential type not exceeding q) of the homogeneous equation obtained from (15) on replacing $\phi(x)$ by 0. If $F(z)$ does not vanish in the region $|z| \leqq q$, the latter solution is identically equal to zero. If $F(z)$ does vanish in this region, then the named solution of the homogeneous equation is identical with the general solution of the differential equation $P(D)u=0$, where D denotes the derivative with respect to x.

It is desirable to give a proof of Theorem 1. The most elegant seems to be the following one, developed by G. B. Lang (in an unpublished Illinois dissertation) on the basis of methods which have been frequently employed by S. Pincherle over a period now stretching to almost fifty years. We present the demonstration merely in outline.

We write $\phi(x)$ and $y(x)$ in the forms

$$\phi(x) = \sum_{\nu=0}^{\infty} s_\nu \frac{x^\nu}{\nu!}, \qquad y(x) = \sum_{\nu=0}^{\infty} t_\nu \frac{x^\nu}{\nu!}.$$

Then we have the conditions

$$\text{(22)} \qquad \limsup_{\nu=\infty} |s_\nu|^{1/\nu} \leqq q, \qquad \limsup_{\nu=\infty} |t_\nu|^{1/\nu} \leqq q.$$

Substituting in (15) and equating coefficients of like powers of x, we have the necessary conditions

$$\text{(23)} \qquad \sum_{\mu=0}^{\infty} a_\mu t_{\mu+\nu} = s_\nu, \qquad (\nu = 0, 1, 2, \cdots).$$

With $\phi(x)$ and $y(x)$ we associate the functions

$$\psi(x) = \sum_{\nu=0}^{\infty} \frac{s_\nu}{x^{\nu+1}}, \qquad g(x) = \sum_{\nu=0}^{\infty} \frac{t_\nu}{x^{\nu+1}}.$$

When $g(x)$ is thus defined, subject only to the second relation in (22), then every function $y(x)$ of exponential type not exceeding q may be expressed in the form

$$\text{(24)} \qquad y(x) = \frac{1}{2\pi i}\int_{C_\rho} g(s)e^{xs}ds,$$

as one may readily verify by aid of the expansion of e^{xs} in powers of s. Since $y(x)$ is to satisfy (15), we must now have conditions (23). By aid of these conditions it may be seen that the Laurent expansion of the function $F(x)g(x)-\psi(x)$ in powers of x contains no non-vanishing terms in negative powers of x.

This suggests that $y(x)$ in (24) be written in the form

$$\text{(25)} \qquad y(x) = \frac{1}{2\pi i}\int_{C_\rho} \frac{\psi(s)}{F(s)} e^{xs}ds + \frac{1}{2\pi i}\int_{C_\rho} \frac{F(s)g(s)-\psi(s)}{F(s)} e^{xs}ds.$$

In the fraction in the second integrand multiply both numerator and denominator by $P(s)/F(s)$, a function which is analytic and does not vanish in the region $|s| \leqq \rho$. Then the last integrand takes the form $A(s)e^{xs}/P(s)$, where $A(s)$ is analytic and single-valued in the region $|s| \leqq \rho$. Therefore the second integral in (25) vanishes when $P(s) \equiv 1$; otherwise the sum of the principal parts of $A(s)/P(s)$ at its poles in the region $|s| \leqq q$ may be written in the form $Q(s)/P(s)$, where $Q(s)$ is a polynomial of degree $n-1$ at most. If $P(s) \equiv 1$, we still write $Q(s)/P(s)$, but understand that $Q(s) \equiv 0$ in this case. Then from (25) we see that the required solution necessarily has the form

$$y(x) = \frac{1}{2\pi i}\int_{C_\rho} \frac{\psi(s)}{F(s)} e^{xs} ds + \frac{1}{2\pi i}\int_{C_\rho} \frac{Q(s)}{P(s)} e^{xs} ds.$$

Since it is easy to verify directly that (20) affords a solution of (15), it follows from the result just obtained that (20) affords the general solution of (1) subject to the conditions which have been imposed.

Several investigators* have contributed to the development of the theory of linear differential equations of infinite order with constant coefficients and of systems of such equations. Several of the memoirs are devoted to the derivation of results very similar to those given in the foregoing theorem; and several methods have been given for their demonstration. Both the methods and the results for the case of equation (15) have been

* See, among others, the following: S. Pincherle, Memorie Reale Accademia delle Scienze, Bologna, (4), vol. 9 (1888), pp. 45–71; C. Bourlet, Annales de l'École Normale Supérieure, (3), vol. 14 (1897), pp. 133–190; S. Pincherle, Mathematische Annalen, vol. 49 (1897), pp. 325–382; T. Lalesco, Journal de Mathématiques, (6), vol. 4 (1908), pp. 125–202; J. F. Ritt, Transactions of this Society, vol. 18 (1917), pp. 21–26, 27–49; F. Schürer, Leipziger Berichte, vol. 70 (1919), pp. 185–246; H. von Koch, Arkiv för Matematik, Astronomi och Fysik, vol. 15, No. 26 (1921), pp. 1–16; S. Pincherle, Acta Mathematica, vol. 48 (1926), pp. 279–304; N. Wiener, Mathematische Annalen, vol. 95 (1926), pp. 557–584; G. Pólya, Göttinger Nachrichten, 1927, pp. 187–195; F. Lettenmeyer, Dissertation, München, 1927, pp. 1–52; I. M. Sheffer, Annals of Mathematics, vol. 30 (1929), pp. 250–264, and Transactions of this Society, vol. 31 (1929), pp. 261–280, 281–289; G. Valiron, Annales de l'École Normale Supérieure, (3), vol. 46 (1929), pp. 25–53; H. T. Davis, American Journal of Mathematics, vol. 52 (1930), pp. 97–108; M. Ghermanesco, Bulletin de l'Académie Royale de Belgique, vol. 19 (1933), pp. 387–394; G. B. Lang, Illinois dissertation, 1935 (unpublished).

extended to systems of such equations, or, more precisely, to systems of the form (4) in which the coefficients $a_{ik\nu}(x)$ are taken to be constants; and this has been carried out for systems having either a finite or an infinite number of equations. This development has, in the main, involved only what is a fairly natural extension of the results for a single equation. Some complications have arisen from the presence of exceptional cases, but it can hardly be said that they have been of a serious character. There has been no important change in the types of the functions appearing in the solutions.

If one notes the character of the functions which have appeared in these solutions, as indicated for instance by our Theorem 1, he may well be struck by the fact that no real novelties have appeared in passing from equations of finite order to equations of infinite order. In fact, in the case of homogeneous equations, the solutions which are set forth by means of the theorem are precisely the same as the solutions of certain related equations of finite order. By allowing the type q of the solution to increase one may indeed bring in more and more of these solutions without altering the given equation of infinite order, but at each stage he will still have only such functions as arise from equations of finite order. By means of the limiting processes which are thus suggested, however, one presumably would be able to introduce new classes of functions, but such functions are not explicit in Theorem 1. This character of the results obtained has persisted throughout a large part (but not all) of the development up to the present. Wherever this character of result persists, it seems fair to say that the theory so developed for differential equations of infinite order has not truly departed from the theory of equations of finite order, whence one would probably be led to suppose that a penetrating understanding of the extended field had not arisen. The pioneer in this more intensive study of differential equations of infinite order is J. F. Ritt.* We turn now to an account of the development due to Ritt and to those who have pursued similar questions further.

It is now convenient to write the equation in the form

$$A(y) \equiv y + a_1 y' + a_2 y'' + \cdots + a_n y^{(n)} + \cdots = 0, \tag{26}$$

* Transactions of this Society, vol. 18 (1917), pp. 27–49.

whence the generating function $F(z)$ has the expansion

(27) $$F(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots .$$

In the investigations now to be considered, the function $F(z)$ is an integral function which is subject to certain additional conditions. It is clear that if a is a zero of $F(z)$ of order μ, then $e^{az}Q(z)$, where $Q(z)$ is a polynomial of degree $\mu - 1$, is a solution of (26). It will be called a fundamental solution.

In his remarkable memoir, Ritt has given (for the first time) general properties of the solution of (26) when the generating function is of genus zero. His results have been extended by G. Pólya* who has shown that, when $F(z)$ is of minimal type of order one, *the analytic solutions of* (26) *are holomorphic functions whose domains of existence are convex.* On the other hand, on making the hypothesis that $F(z)$ has only a finite number of multiple zeros and that the moduli of its zeros are sufficiently regular,† Ritt has shown that any solution whatever may be developed in a series of fundamental solutions valid in the whole domain of its existence.

By means of this result, Ritt gave for the first time the extension of the Fabry and Hadamard theorem on lacunary Taylor series to the case of series of the form

$$\sum c_n e^{\alpha_n z}.$$

If the constants $\alpha_1, \alpha_2, \cdots$ *satisfy the stated regularity condition, then the frontier of the domain of convergence of this series* (*a domain which is convex*) *is an essential cut for the function which it defines.* The demonstration of this theorem given by Ritt does not differ in any essential way from that which was independently given later by Landau and Carlson.

The later work of G. Valiron‡ on linear differential equations of infinite order with constant coefficients contains what seems to be the most important results yet developed along the lines

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* Göttinger Nachrichten, 1927, pp. 187–195.

† If the zeros of $F(z)$ are denoted by $\alpha_1, \alpha_2, \alpha_3, \cdots$, then the regularity condition imposed by Ritt may be stated as follows. There exists an integer r such that for $n \geqq r$ we have $|\alpha_{n+1}/\alpha_n| > 1 + k/n$, where k is a suitable constant greater than 2. This condition is used by Ritt for the proof of certain needful properties of the generating function $F(z)$.

‡ Annales de l'École Normale Supérieure, (3), vol. 46 (1929), pp. 25–53.

of Ritt's path-breaking contribution. The work of Pólya had appeared in the meantime. Valiron set himself the problem of extending the results of Ritt. He removes the hypothesis that the number of multiple zeros of $F(z)$ is limited. Moreover his hypotheses on $F(z)$ are in other ways less restrictive than those of Ritt. He finds that all the results of Ritt remain valid if one supposes only that $F(z)$ is of minimal type of order one (a fact which Valiron communicated to Ritt before the appearance of Pólya's memoir). Some modifications of the results are necessary when $F(z)$ is of mean type of order one. The memoir of Valiron follows rather closely the methods of Ritt. It is rich in results other than those which we have explicitly indicated.

These investigations by Ritt and by Pólya and by Valiron mark the beginning of an intensive investigation of differential equations of infinite order along lines which bring to light essential features which are characteristic of the fact that the order of the equation is infinite. It seems clear that results of this character should be in the forefront of attention.

We shall close this section by stating without proof a hitherto unpublished general theorem from which a number of previously known results are readily obtained, but unfortunately none of those deep-lying theorems which Ritt and his followers have obtained. This theorem belongs to a range of ideas which have been especially emphasized by Pincherle.

Consider the equation (15). Let $F(z)$, as in (19), be the corresponding characteristic function. We suppose that $F(z)$ and $\phi(x)$ are both integral functions, and we write $\phi^{(\nu)}(0)=s_\nu$. The solution $y(x)$ of (15) is required to be an integral function. Let τ be a given positive number. Let $\{\lambda_\nu\}$ and $\{\mu_\nu\}$, $(\nu=0, 1, 2, \cdots)$, be two infinite sequences of positive numbers such that $\tau \leqq \mu_\nu \leqq \lambda_\nu$ for every ν. For each particular value of ν let C_ν be a closed contour of finite length encircling the point 0 and lying in the ring $\mu_\nu \leqq |z| \leqq \lambda_\nu$, and let C_ν pass through no point at which $F(z)$ vanishes. Define T_ν by the relation

$$T_\nu = \sum_{k=0}^{\infty} | a_k | \lambda_\nu^k .$$

Then we have the following result.

A sufficient condition that $y(x)$, defined by the relation

$$y(x) = \sum_{\nu=0}^{\infty} \frac{s_\nu}{2\pi i} \int_{C_\nu} \frac{e^{xz} dt}{z^{\nu+1} F(z)},$$

shall be an integral function satisfying equation (15) *is that the series*

$$\sum_{\nu=0}^{\infty} e^{\rho\lambda_\nu} \mid s_\nu \mid T_\nu \mu_\nu^{-\nu-1} \int_{C_\nu} \frac{\mid dz \mid}{\mid F(z) \mid}$$

shall be convergent for every positive number ρ.

5. *Equations with Polynomial Coefficients.* The typical equation here is of the form

$$(28) \qquad \sum_{n=0}^{\infty} (a_{n0} + a_{n1}x + a_{n2}x^2 + \cdots + a_{np}x^p) y^{(n)}(x) = \phi(x),$$

where p is a positive integer and not all the quantities a_{np} are zero. This is the special case of equation (1) in which the coefficients $a_k(x)$ are all polynomials of bounded degree. Corresponding systems (2) and (4) and the generalizations of them indicated in §2 also come into play here, the coefficients being restricted in each case to be polynomials of bounded degree. Such equations and systems of equations have formed the subjects of a considerable number of investigations.*

So far as I am aware, an intensive investigation of these equations along the line of what may be called the Ritt tradition has never been carried out. In fact I know of no effective pursuit of such questions other than those which belong to equations with constant coefficients. So far as the subject of the present section is concerned, the investigations up to date correspond

* See among others, the following: S. Pincherle, Memorie Accademia delle Scienze, Ist., Bologna, (4), vol. 9 (1888), pp. 181–204; C. Bourlet, Annales de l'École Normale Supérieure, (3), vol. 14 (1897), pp. 133–190; T. Lalesco, Journal de Mathématiques, (6), vol. 4 (1908), pp. 125–202; F. Schürer, Leipziger Berichte, vol. 70 (1919), pp. 185–240; E. Hilb, Mathematische Annalen, vol. 82 (1920), pp. 1–39; E. Hilb, Mathematische Annalen, vol. 84 (1921), pp. 16–30, 43–52; O. Perron, Mathematische Annalen, vol. 84 (1921), pp. 1–15, 31–42; H. von Koch, Arkiv för Matematik, Astronomi och Fysik, vol. 16 (1922), No. 6, 21 pp.; F. Lettenmeyer, Dissertation, München, 1927, pp. 1–52; I. M. Sheffer, Annals of Mathematics, vol. 30 (1929), pp. 345–372; I. M. Sheffer, Transactions of this Society, vol. 31 (1929), pp. 261–280; H. T. Davis, Annals of Mathematics, vol. 32 (1931), pp. 686–714.

in the main to aspects of the problem similar to those treated in our §4 exclusive of what has arisen from the use of the methods of Ritt. Even for these relatively simple questions concerning equation (28) no inconsiderable difficulties have been encountered.

Pincherle's detailed investigation, in the paper cited, was confined to a certain generalized difference equation with rational coefficients; but the methods employed by him, extending those which he had previously used for the case of constant coefficients, have played an important role in the subsequent development of the theory; and this is the reason for our calling attention to his work in the present connection. In several papers, not mentioned in our references, some other aspects of the theory of difference equations have been considered in such a way as to throw light on the theory of differential equations of infinite order.

It seems that the first explicit treatment of equation (28) is that contained in Lalesco's memoir of 1908, though results concerning it were implicit in certain earlier papers, particularly those of Bourlet and Pincherle both of whom had previously published several papers bearing indirectly on the problem. Lalesco, incidental to a consideration of the inversion of Volterra integrals, applied to the homogeneous equation the general Laplace transformation

$$y(x) = \int_L e^{xt}v(t)dt,$$

choosing the path of integration conveniently with reference to the coefficients in the equation. In 1920 and 1921 E. Hilb developed the method in more detail.

O. Perron (loc. cit., 1921) avoided the use of the Laplace transformation and the general theory of infinite matrices and gave a quite elementary development by means of the theory of systems of equations of the form

$$\sum_{n=0}^{\infty} (\alpha_n + \beta_{mn})x_{m+n} = c_m, \qquad (m = 0, 1, 2, \cdots).$$

He assumed that $\phi(x)$ is of exponential type not exceeding q and that the $a_{\nu\lambda}$ are such that the functions

$$h_\lambda(z) = \sum_{\nu=0}^{\infty} a_{\nu\lambda} z^\nu, \qquad (\lambda = 0, 1, \cdots, p),$$

are regular for $|z| \leqq q$ (whence the convergence radii of these series are greater than q). Then he showed that the homogeneous equation, that for which $\phi(x) \equiv 0$ in (28), has exactly $\rho - p + s$ linearly independent solutions of exponential type not exceeding q, where ρ is the number of zeros of $h_p(z)$ in the circle $|z| \leqq q$ (multiple zeros counted multiply), and where s denotes the number of linearly independent solutions, regular in the region $|z| \leqq q$, of the auxiliary equation

$$\sum_{\lambda=0}^{p} h_\lambda(z) u^{(\lambda)}(z) = 0$$

of order p. He showed further that the differential equation (28) has, for every choice of the function $\phi(x)$ of exponential type not exceeding q, solutions y of exponential type not exceeding q when and only when the corresponding homogeneous equation has exactly $\rho - p$ integrals.

Lettenmeyer (loc. cit., 1927) extended the methods of Perron to systems of equations having polynomial coefficients of bounded degree.

E. Hilb (loc. cit., 1920, 1921) investigated equation (28) by aid of the infinite set of equations in an infinite number of unknowns obtained through unlimited differentiation of the equation. He obtained conditions assuring the uniqueness of a solution.

While his results were still unpublished Hilb communicated some of them to H. von Koch. These turned the latter's attention again to some ideas with which he had been concerned in 1912 and it turned out that they were capable of leading to elegant methods for dealing with equations (28) and also with the corresponding equations with constant coefficients. Thus, almost simultaneously, three different authors, under the influence of widely diverse guiding ideas, were investigating equations of the form (28). The work of F. Schürer in 1919 also bears on the same problem.

In 1929 I. M. Sheffer dealt with equations with constant coefficients and with linear coefficients. He also investigated expansions in generalized Appell polynomials and treated certain related linear functional equations. The methods employed are similar to those which have been used by Pincherle for many years in the investigation of related problems. They arise essentially from the use of the Laplace transformation and the reduction of the equation to a contour integral equation by means of which the investigation may be carried forward. The method is extended to partial differential equations and to "Laurent differential equations," both with constant coefficients.

H. T. Davis (loc. cit., 1931) treated equation (28) by means of the calculus of operators, employing the symbolic methods used by Pincherle and especially by Bourlet. The formal solution of (28) thus obtained is reduced to three useful forms. There is a discussion of the validity of the formal solutions and the domain of functions to which the operators apply, the domain being extended beyond that considered by previous writers. "The difficulties admitted by this extension," the author adds, "have not been entirely resolved, however, since they have been discovered to be inherent in the nature of asymptotic and summable series, the theory of which is still obscure in many points."

On this note of inachieved results we take our leave of the important subject treated in this section, adding merely the statement of our judgment that the results attained up to the present have hardly penetrated the surface. The most characteristic elements of this theory, it would seem, still await elucidation in the future.

6. *Equations with Analytic Coefficients.* Much of the work of Pincherle and Bourlet, referred to in earlier sections, has important bearings on equations with analytic coefficients; we shall, however, omit further analysis of these contributions, confining our attention (in this section) to certain other papers which bear more specifically on the analytic function-theoretic aspects of the theory of linear differential equations of infinite order.

H. von Koch* considered a system of the form

$$\frac{dx_i}{dt} = \sum_{\nu=1}^{\infty} \alpha_{i\nu} x_\nu, \qquad (i = 1, 2, \cdots), \tag{29}$$

*Öfvers. Kongl. Svenska Vetenskaps-Akademien, Handlingar, (Stockholm), vol. 56 (1899), pp. 395–411.

where the $\alpha_{i\nu}$ denote power series in t which converge when $|t|<R$, and where $|\alpha_{i\nu}|<S_iA_\nu$ in the region $|t|\leqq\rho<R$, S_i and A_ν being independent of T and being such that the series $\sum S_kA_k$ converges. Let $x_1^{(0)}, x_2^{(0)}, \cdots$ be a sequence of constants such that the series $\sum x_k^{(0)}A_k$ converges absolutely. Then it is shown that there is one and only one integral system of power series $x_1, x_2, x_3, \cdots$ which for $t=0$ take respectively the initial values $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \cdots$. These series converge when $|t|<R$. Emphasis is placed on the fact that the notion of fundamental systems of solutions of (1) may be introduced and that a significant part of the Fuchs theory of linear differential equations may be carried over to these systems. Extensions of the theory are also presented.

Lalesco's paper of 1908, cited in §4, contains results relevant to the present section.

M. Gramegna* employed the matrix notation and certain symbolic processes in treating (29) and the corresponding non-homogeneous system.

F. R. Moulton† treated the infinite system of differential equations

$$(30)\qquad \frac{dx_i}{dt} = f_i(t;\, x_1, x_2, \cdots) = a_i + f_i^{(1)} + f_i^{(2)} + \cdots, \qquad (i = 1, 2, \cdots),$$

where a_i is a constant and $f_i^{(j)}$ is the totality of terms in the power series expansion of f_i which are homogeneous in $t, x_1, x_2, \cdots$ and of degree j. The functions f_i are said to be of analytic type. It is assumed that the following hypotheses are satisfied:

(H_1) $x_i=0$ at $t=0$ for each i of the set $1, 2, 3, \cdots$;

(H_2) Finite real positive constants $c_0, c_1, c_2, \cdots, r_0, r_1, r_2, \cdots$. A, a exist such that

$$(31)\qquad s = c_0t + c_1x_1 + c_2x_2 + \cdots$$

converges if

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* Atti Reale Accademia delle Scienze, Torino, vol. 45 (1910), pp. 469–491.

† Proceedings National Academy of Sciences, vol. 1 (1915), pp. 350–354. The substance of this investigation, together with some applications, is contained in the final chapter of Moulton's book on *Differential Equations*, 1930.

$$(32) \qquad |t| \leqq r_0, \qquad |x_i| \leqq r_i, \qquad (i = 1, 2, \cdots),$$

and such that Ar_is^j dominates $f_i^{(j)}$ and $a_i \leqq Ar_ia$.

If an analytic solution of (30) exists, satisfying the initial conditions (H_1), it will necessarily have the form

$$x_i = A_i^{(1)}t + A_i^{(2)}t^2 + A_i^{(3)}t^3 + \cdots, \qquad (i = 1, 2, \cdots).$$

It is readily shown that just one formal solution of this form exists. In order to prove the suitable convergence of the series in this formal solution, and hence to establish the existence of an actual solution, the author employed the method of dominant functions in much the same way as that which is usual in the corresponding case of finite systems. The result obtained is a natural generalization of the classic theory of finite systems.

The limitations placed on t in carrying out the convergence proof are so restrictive that the corresponding x_i are not shown to attain the boundary of the region for which the right members of (30) converge. A question thus arises whether the solution can be continued beyond the domain indicated by the first argument. That the answer is affirmative is shown by the presentation of an effective method for obtaining the desired continuation.

W. G. Simon* proved general existence theorems for equations of the form (29) and in particular treated certain types of solutions of particular kinds of systems with periodic coefficients. He found that many of the phenomena of the finite systems are carried over into the infinite systems of differential equations.

W. Sternberg† treated the system

$$(33) \qquad \frac{dy_i}{dx} = a_{i0}(x) + \sum_{j=1}^{\infty} a_{ij}(x)y_j, \qquad (i = 1, 2, \cdots),$$

where the $a_{ik}(x)$ are analytic functions of x which are regular in the region S defined by the relation $|x-a| \leqq r$, where a is a given point and r is a given positive constant. It is assumed that the series in the relations

$$g_i(x) = |a_{i0}(x)| + |a_{i1}(x)| + |a_{i2}(x)| + \cdots, \quad (i = 1, 2, \cdots),$$

* American Journal of Mathematics, vol. 42 (1920), pp. 27–46.

† Heidelberg Akademie Sitzungsberichte, 1920, No. 10, pp. 1–21.

all converge uniformly in S. Then constants N_i exist such that we have $0 \leqq g_i(x) \leqq N_i$, $(i=1, 2, \cdots)$, throughout S. It is assumed that N exists such that $N_i \leqq N$, $(i=1, 2, \cdots)$. Let $b_1, b_2, \cdots$ be any set whatever of constants such that $|b_i| < C$, $(i=1, 2, \cdots)$, for some appropriate constant C. Then it is shown that there is one and only one system of integrals $y_1, y_2, \cdots$ of (33) which satisfy the initial conditions $y_i|^{x=a} = b_i$ and are regular throughout S. These integrals are bounded functions of x and throughout S satisfy for each i the inequality $|y_i(x)| < Ce^{Nr}$. The proof is made by the method of successive approximations.

The corresponding homogeneous system (namely, that for which $a_{i0}(x) \equiv 0$) is treated under the additional hypothesis that constants K and n, $n>1$, exist such that $|a_{ik}(x)| < K(ik)^{-n}$ in S, and it is shown that systems of solutions exist having the characteristic properties of fundamental systems. It is finally indicated that the essential results hold under somewhat less restrictive hypotheses.

It is clear that we have here (as in previous cases) a choice of hypotheses which lead to conclusions as nearly similar as possible to those subsisting in the finite case. What one would prefer to see are results which are characteristic of the infinite case. But these, in the main, await discovery in the future.

Infinite systems of differential equations have also been treated by A. Wintner,* D. C. Lewis,† and others. (See also the next section.)

Let us consider the (finite or infinite) differential operator $P(D)$ defined by the relation

$$(34) \qquad P(D) = a_0D^0 + a_1D + a_2D^2 + a_3D^3 + \cdots,$$

where $a_0, a_1, a_2, \cdots$ are all given functions of x and where D is the symbol for differentiation with respect to x. The theory of linear differential equations of infinite order has brought to notice those operators $P(D)$ which have the following general property in common with the operation of differentiation, namely, that for any given finite point whatever x_0 and any

* Mathematische Annalen, vol. 95 (1926), pp. 544–556; vol. 98 (1928), pp. 273–280; American Journal of Mathematics, vol. 53 (1931), pp. 241–257.

† Transactions of this Society, vol. 35 (1933), pp. 792–823.

given function $u(x)$ whatever, analytic at x_0, the operator $P(D)$ shall be applicable to $u(x)$ and shall yield a resulting function

$$P(D)u \equiv a_0 u + a_1 u' + a_2 u'' + \cdots$$

which is itself analytic at x_0. By considering the functions $u(x) = (x - x_0)^k$, for varying points x_0 and varying non-negative integers k, it is easy to show that each of the functions $a_0, a_1, a_2, \cdots$ is an integral function.

In fact it is not difficult to show that for the required conditions on $P(D)$ it is necessary that the functions $a_0, a_1, a_2, \cdots$ shall be integral functions which for every x_0 verify the relation

$$\lim_{\nu=\infty} \left| a_\nu(x_0)\nu! \right|^{1/\nu} = 0.$$

On the other hand, it may be shown to be sufficient that the functions $a_0, a_1, a_2, \cdots$ shall be integral functions which for every non-negative number σ verify the relation

$$\lim_{\nu=\infty} (M_{\nu\sigma}\nu!)^{1/\nu} = 0,$$

where $M_{\nu\sigma}$ is the maximum value of $|a_\nu(x)|$ for $|x| = \sigma$.

Whether the necessary condition and the sufficient condition here given can be brought closer together by simple means I have not determined, even though the results as they stand are not altogether satisfactory. The reader will not find it difficult to supply the proof for at least as much as is stated here.

7. *Some Further Problems and Connections.* The problem of an infinite system of differential equations was treated by E. H. Moore* at the fourth international congress of mathematicians at Rome in 1908, from the point of view of general analysis, the functions not being restricted to those of analytic type. In a series of papers W. L. Hart† developed theorems concerning a type of real-valued functions of infinitely many real variables and employs these results in treating infinite systems of ordinary differential equations both non-linear and linear; the linear

* Atti del IV Congresso Internazionale Matematica, vol. II (1909), pp. 98–114.

† Proceedings National Academy of Sciences, vol. 2 (1916), pp. 309–313; Transactions of this Society, vol. 18 (1917), pp. 125–160; American Journal of Mathematics, vol. 39 (1917), pp. 407–424.

system is associated with its adjoint, and the notion of fundamental systems of solutions is employed. T. H. Hildebrandt* developed a theory of linear differential equations in general analysis, utilizing both the concepts of general analysis and the general theory of integral equations. P. Flamant† has applied the method of successive approximations to equations of the form (33), pointing out that the results are valid both for the case of analytic functions and for that of continuous functions of a real variable.

W. L. Hart‡ developed the Cauchy-Lipschitz method for infinite systems of differential equations. I. A. Barnett§ treated both ordinary and partial differential equations with a continuous infinitude of variables. H. T. Davis|| has associated fractional operations with problems of the type here treated. L. Pomey¶ has published a series of relevant papers. N. Wiener** has treated the operational calculus. W. T. Reid†† has developed a theory of infinite systems with auxiliary boundary conditions; he has also treated infinite systems in the domain of Lebesgue summable functions. Finally, there are many other papers dealing with our subject; they are written both by the authors here quoted and by others; considerations of space prevent our treating them; for the most part they will come to the reader's attention through references given in the papers referred to in this address.

There are several aspects of the theory of difference equations of finite order which illuminate and are illuminated by the theory of differential equations of infinite order, as will be seen from the relevant papers of Pincherle, Hilb, Perron, Carmichael,

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* Transactions of this Society, vol. 18 (1917), pp. 73–96, and vol. 19 (1918), pp. 97–108.

† Bulletin des Sciences Mathématiques, (2), vol. 45 (1921), pp. 81–87.

‡ American Journal of Mathematics, vol. 43 (1921), pp. 226–231.

§ American Journal of Mathematics, vol. 44 (1922), pp. 172–190, and vol. 45 (1923), pp. 42–53.

|| American Journal of Mathematics, vol. 46 (1924), pp. 95–109, and vol. 49 (1927), pp. 123–142.

¶ Paris Comptes Rendus 1926, 1927, 1928, and elsewhere.

** Mathematische Annalen, vol. 95 (1926), pp. 557–584.

†† Transactions of this Society, vol. 32 (1930), pp. 284–318; Annals of Mathematics, vol. 32 (1931), pp. 37–46.

Bochner, Ghermanesco, and others; but we cannot here treat them further than the bare indication given in §3. It may be mentioned, however, that some of the theorems about differential equations of infinite order imply corresponding results about difference equations of infinite order, but ordinarily they do not yield the latter results under the most natural hypotheses. The suggestion is inevitable that we are in need of a more direct theory of difference equations of infinite order. A few results looking in this direction appear in the dissertation of G. B. Lang (already mentioned). Integral equations and integro-differential equations have been associated with differential equations of infinite order. In this connection one may mention an important paper by F. Schürer* on a common method of treating certain problems involving functional equations.

Problems relating to expansions in Appell polynomials are intimately connected with the theory of linear differential equations of infinite order with constant coefficients, a fact which was clearly recognized by Pincherle about fifty years ago. Recently, I. M. Sheffer† has developed a theory of expansions in generalized Appell polynomials‡ and has applied the results to a class of related linear functional equations. He was primarily concerned with linear differential equations of infinite order with polynomial coefficients of bounded degree and their relation to expansions in generalized Appell polynomials. Let

$$A_i(t) \sim \sum_{n=0}^{\infty} \alpha_{in} t^n, \quad (i = 0, 1, 2, \cdots, k),$$

be $k+1$ formal power series, with $A_k(t) \neq 0$. Then the generalized Appell polynomials $\{G_n(x)\}$, of order k, are defined by the (formal) expansion

$$e^{tx}\{A_0(t) + xA_1(t) + \cdots + x^k A_k(t)\} \sim \sum_{n=0}^{\infty} G_n(x) t^n.$$

A general theory is given for the expansion of functions $f(x)$ of exponential type in terms of the polynomials $G_n(x)$, this theory being developed in intimate connection with differential equa-

* Leipziger Berichte, vol. 70 (1919), pp. 185–240.

† Transactions of this Society, vol. 31 (1929), pp. 261–280.

‡ Pincherle also considered certain generalized Appell polynomials.

tions of the type mentioned earlier in this paragraph. By playing the two theories one against the other the author makes useful extensions of both of them. Related expansion problems are likely to undergo a considerable development in the near future.

We may fittingly bring these remarks to a close by emphasizing one aspect of the existing theory of differential equations of infinite order to which we have already directed attention. In the main it has been true that the results mentioned are closely analogous to corresponding ones for differential equations of finite order. The theorems for the most part take the form which is suggested by corresponding theorems for the simpler case. In fact, there often seems to be a conscious purpose to formulate the hypotheses in such a way as to preserve the analogies in the most intimate form possible. This is perhaps to be expected in a first approach to the more general subject; it is natural to inquire to what extent analogies subsist. Again, in so far as explicit information about the functions in a solution is concerned, we usually have only that which arises in close analogy with the finite case. In fact, in all our remarks (except for differences shown by examples) it has been true, with one single exception, that the results presented indicate no marked departure from the finite case. So far as I am aware, there is no other general exception to this statement to be found in the literature. The exception to which we refer is in the results initiated and inspired by J. F. Ritt, as already indicated in our §4. In these results of Ritt and Pólya and Valiron we have properties of the solutions of linear differential equations of infinite order which are different from anything that arises in the finite case. They are characteristic of the equation with respect to its being of infinite order. It would seem to be beyond dispute that such characteristic properties are the ones most eagerly to be sought. And yet a single small group of them stands out as apparently unique in the whole literature. This fact presents a challenge to analysts interested in functional equations. We have hardly penetrated the surface of the important theory of differential equations of infinite order. We need a further development of those elements of the theory of these equations which are characteristic of them as being of infinite order.

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