A NOTE ON LIPSCHITZ CLASSES

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This note consists in the application of some results of Hardy and Littlewood* on fractional integrals to a theorem of Paley and Zygmund† and gives a generalization of that theorem.

We consider only functions of the Fourier power series type. That is, $f(x)$ is periodic in 2π, integrable, and with a Fourier series of the form

$$f(x) \backsim \sum_{n=0}^{\infty} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

In dealing with functions of the class $\mathrm{Lip}(\alpha)$ or $\mathrm{Lip}(\alpha, p)$, $\alpha \neq 1$, this restriction is a matter of convenience rather than one of necessity.‡

A function $f(x)$ is said to belong to the class $\mathrm{Lip}(\alpha)$, where $0 \leqq \alpha \leqq 1$, in the interval $(-\pi, \pi)$, if

$$f(x+h) - f(x-h) = O(h^{\alpha})$$

uniformly for $-\pi \leqq x-h < x+h \leqq \pi$, and to $\mathrm{Lip}(\alpha, p)$, where $p \geqq 1$, $0 \leqq \alpha \leqq 1$, in $(-\pi, \pi)$, if $f(x) \epsilon L$, and

$$\int_{-\pi}^{\pi} | f(x+h) - f(x-h) |^{p} dx = O(h^{\alpha p}).$$

The functions $\phi_n(t)$, $(n = 0, 1, 2, \cdots)$, are the Rademacher functions.§

* Hardy and Littlewood, *Some properties of fractional integrals* I, Mathematische Zeitschrift, vol. 27 (1927–28), pp. 565–606. We will refer to this paper as (HL).

† Paley and Zygmund, *On some series of functions*, Proceedings Cambridge Philosophical Society, vol. 26 (1930), pp. 337–357. A. Zygmund, *Trigonometrical Series*, 1935, §5.61. We will refer to this book as (Z). It contains extensive bibliographical references.

‡ Hardy and Littlewood, *A convergence criterion for Fourier series*, Mathematische Zeitschrift, vol. 28 (1928), pp. 612–634, in particular, §2 and §3.5. See also (Z), §7.4.

§ For definitions and properties see (Z), §§1.32 and 5.5 to 5.61.

THEOREM 1. *Let $c_0, c_1, c_2, \cdots, c_n, \cdots$ be a sequence of real or complex numbers such that*

$$\sum_{n=2}^{\infty} n^{2\alpha} \left| c_n \right|^2 (\log n)^{1+\epsilon}$$

converges for an $\epsilon > 0$. Then, for almost all values of t, the function

$$f_t(x) \sim \sum_{n=0}^{\infty} c_n e^{inx} \phi_n(t)$$

belongs to the class Lip(α), $(0 \leqq \alpha \leqq 1)$. *The theorem is false for the case $\alpha = 0$ or $\alpha = 1$ if $\epsilon = 0$.*

As a consequence of the theorem of Paley and Zygmund mentioned above it follows that

$$f_t^{\alpha}(x) = i^{\alpha} \sum_{n=1}^{\infty} n^{\alpha} c_n e^{inx} \phi_n(t),$$

for almost all values of t, is a continuous function (since the series converges uniformly we put $f_t^{\alpha}(x)$ equal to the sum of the series). That is, we have

$$f_t^{\alpha}(x) \epsilon \text{ Lip } (0).$$

If by the symbol $f_{t,\alpha}^{\alpha}(x)$ we denote the integral of $f_t^{\alpha}(x)$ of order α, we have*

$$f_{t,\alpha}^{\alpha}(x) \epsilon \text{ Lip } (\alpha).$$

But

$$f_{t,\alpha}^{\alpha}(x) = i^{\alpha} \sum_{n=1}^{\infty} \frac{c_n n^{\alpha}}{(in)^{\alpha}} e^{inx} \phi_n(t)$$
$$= f_t(x) - c_0 \phi_0(t).$$

To show that the theorem is not true in the case $\alpha = 1$, for $\epsilon = 0$, we consider the function

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* (Z), §§9.80 and 9.81. A function satisfies a condition Lip*(α) or Lip*(α, p) when it satisfies a condition analogous to that for Lip(α) or Lip(α, p) but with o small in place of O large. In each of our theorems Lip(α) or Lip(α, p) may be replaced by Lip*(α) or Lip*(α, p), respectively, except in the case $\alpha = 1$; this follows from Theorems 18, 21, and 22 of (HL).

$$\sum_{m=1}^{\infty} \frac{\pm\, i e^{i2^m x}}{2^m m \log (m+1)}.$$

This can not belong to Lip(1) for any sequence of signs since it is the integral of

$$\sum_{m=1}^{\infty} \frac{\pm\, e^{i2^m x}}{m \log (m+1)},$$

which is Paley and Zygmund's example* of a series which does not represent a bounded function for any sequence of signs.

For the case of Lip (α, p), $(0 \leqq \alpha \leqq 1,\ p \geqq 1)$, we have a similar theorem.

THEOREM 2. *Let* $c_0, c_1, c_2, \cdots, c_n, \cdots$ *be a sequence of real or complex numbers such that* $\sum_{n=1}^{\infty} n^{2\alpha} |c_n|^2$ *converges. Then, for almost all values of* t, *the function*

$$f_t(x) \backsim \sum_{n=0}^{\infty} c_n e^{inx} \phi_n(t)$$

belongs to the class Lip (α, p), $(p \geqq 1,\ 0 \leqq \alpha \leqq 1)$.

Since $\sum_{n=1}^{\infty} |n^\alpha c_n|^2$ is convergent, it follows that†

$$f_t^{\alpha}(x) \backsim i^\alpha \sum_{n=1}^{\infty} n^\alpha c_n e^{inx} \phi_n(t)$$

belongs to L_p, $(p \geqq 1)$.

Now by a theorem of Hardy and Littlewood‡

$$f_{t,\alpha}^{\alpha}(x) \epsilon \text{ Lip } (\alpha, p).$$

But, as before,

$$f_{t,\alpha}^{\alpha}(x) = f_t(x) - c_0 \phi_0(t).$$

As a corollary of the following theorem we have a better theorem for the case $1 \leqq p \leqq 2$.

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* Paley and Zygmund, loc. cit., p. 350. This gives the case $\alpha = 0$, $\epsilon = 0$.

† (Z), §5.6 (iii).

‡ (HL), Theorems 21, 22, and ff.

THEOREM 3. *If*

$$\sum_{n=-\infty}^{+\infty} n^{p'\alpha} | c_n |^{p'}, \qquad (1 < p' \leqq 2),$$

converges, then

$$f(x) \backsim \sum_{n=-\infty}^{+\infty} c_n e^{in\alpha}$$

belongs to the class Lip (α, p), $1/p+1/p'=1$, $(0 \leqq \alpha \leqq 1)$.

From the Young-Hausdorff* theorem we have

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} | f^{\alpha}(x) |^p dx\right)^{1/p} \leqq \left(\sum_{n=-\infty}^{+\infty} | n^{\alpha} c_n |^{p'}\right)^{1/p'},$$

where

$$f^{\alpha}(x) \backsim i^{\alpha} \sum_{n=-\infty}^{+\infty} n^{\alpha} c_n e^{inx}.$$

Since $f^{\alpha}(x) \epsilon L_p$, we have†

$$f(x) - c_0 = f_{\alpha}^{\alpha}(x) = \sum_{n=-\infty}^{+\infty}{}' c_n e^{inx} \epsilon \text{ Lip } (\alpha, p), \quad (p \geqq 2).$$

COROLLARY. *If*

$$\sum_{n=-\infty}^{+\infty} n^{2\alpha} | c_n |^2, \qquad (0 \leqq \alpha \leqq 1),$$

converges, then

$$f(x) \backsim \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

belongs to Lip (α, p) *for every* p *such that* $1 \leqq p \leqq 2$.

This follows because

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} | f(x) |^p dx\right)^{1/p} \leqq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} | f(x) |^2 dx\right)^{1/2}.$$

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* (Z), §9.1.

† (HL), Theorem 21.