DISTRIBUTION OF MASS FOR AVERAGES OF NEWTONIAN POTENTIAL FUNCTIONS

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1. *Introduction.* It has been proved that the average of a potential function over a spherical volume and the average of a potential function over a spherical surface are themselves potential functions.* This paper is concerned with the determination of the distribution of mass for these two spherical averages; in addition, the distribution of mass for more general averages is obtained.

2. *Preliminary Theorems.* The problem is solved by means of a theorem on the change of the order of integration of an iterated Stieltjes integral. First it is necessary to state some preliminary theorems. We recall the following elementary theorem.

If $h(Q)$ is continuous in Q and $g(e)$ is a distribution of positive mass, bounded in total amount, and lying on a bounded set F (which may be taken as closed without loss of generality), then, for the integral over the whole of space, w,

$$\left| \int_w h(Q)dg(e_Q) - \sum_w h(Q_i)g(e_i) \right| < \omega_\delta \alpha, \tag{1}$$

where the summation is extended over all the meshes of a lattice L_δ, of diameter $\leqq \delta$, Q_i is a point of the mesh e_i, ω_δ is the oscillation of $h(Q)$ on a subset of F of diameter $\leqq \delta$, and $\alpha \geqq g(F)$.

This theorem will be applied to the integral

$$\int_w h^N(M, Q)dg(e_Q, P),$$

where $h^N(M, Q)$ is continuous in M, Q, and $g(e, P)$ and F are bounded independently of P, so that ω_δ and α in (1) are independent of M, P.

THEOREM 1. *If $g(e, P)$ is a distribution of positive mass, bounded independently of P, on a set F bounded independently*

* G. C. Evans, *On potentials of positive mass*, Transactions of this Society, vol. 37 (1935), p. 250.

of P, and if $f(e)$ is a distribution of positive mass, bounded in all space, and if $g(e, P)$ is summable with respect to $f(e)$, then

$$G(e) = \int_w g(e, P)df(e_P)$$

is a distribution of positive mass bounded in total amount and lying on the set F.

In this theorem, it is not required that $g(e, P)$ be continuous in P, so that the integral must be interpreted as a generalized or Daniell integral with respect to $f(e)$. But in the theorems which follow, $g(e, P)$ is taken as continuous in P for every mesh of L.

To prove that $G(e)$ is a distribution of positive mass, we must show that $G(e)$ is a non-negative, absolutely additive function of point sets (only Borel measurable point sets are considered).

The first requirement follows immediately from the fact that $g(e, P)$ and $f(e)$ are non-negative for all P and e. In order to exhibit the second condition, let $e = e_1 + e_2 + \cdots$ and $e_n' = e_1 + e_2 + \cdots + e_n$; as $g(e, P)$ is a distribution of positive mass, we have

$$g(e, P) = \lim_{n=\infty} g(e_n', P), \text{ with } g(e_{n+1}', P) \geqq g(e_n', P).$$

Since $g(e, P)$ is, as a function of P, the limit of an increasing sequence of functions, the order of integration and passing to the limit may be interchanged; so that

$$G(e) = \int_w g(e, P)df(e_P) = \lim_{n=\infty} \int_w g(e_n', P)df(e_P)$$
$$= \lim_{n=\infty} [G(e_1) + \cdots + G(e_n)] = G(e_1) + G(e_2) + \cdots.$$

Hence $G(e)$ is a distribution of positive mass. Also $G(e)$ is a distribution of mass lying on F; that is, $G(e) = 0$, if $e \cdot F = 0$, because of the hypothesis that $g(e, P) = 0$, if $e \cdot F = 0$. Finally, $G(e) \leqq$ [upper bound of $g(e, P)$] $\cdot f(w)$, so bounded.

We state the generalizations of two theorems proved by H. E. Bray; the proofs are omitted as they are essentially the same as those given by Bray.* They depend on the inequality (1).

* H. E. Bray, *Elementary properties of the Stieltjes integral*, Annals of Mathematics, vol. 20 (1918–19), pp. 180–185.

THEOREM 2. *If $h^N(M, Q)$ is a continuous function of M and Q, and is bounded in all space, if $g(e, P)$ is a distribution of positive mass, bounded independently of P, on a set F bounded independently of P, and if $g(e, P)$ is continuous in P for every cell e of a net L, then*

$$K(M, P) = \int_w h^N(M, Q)dg(e_Q, P)$$

is continuous in M and P.

THEOREM 3. *If $h^N(M, Q)$ is continuous in M and Q, and bounded in all space, if $g(e, P)$ is a distribution of positive mass, bounded independently of P, on a set F bounded independently of P, and if it is continuous in P for every cell e of a net L, and if $f(e)$ is a distribution of positive mass, bounded in total amount, lying on a bounded set E (which may be taken as closed without loss of generality), then the integrals*

$$\int_w df(e_P) \int_w h^N(M, Q)dg(e_Q, P),$$

and

$$\int_w h^N(M,Q)d\left[\int_w g(e_Q, P)df(e_P)\right]$$

exist and are equal.

We may now state and prove the concluding theorem in this series.

THEOREM 4. *If $g(e, P)$ is a positive distribution of mass, bounded in total amount independently of P, on a set F bounded independently of P, and if it is continuous in P for every cell e of the net L, and if $f(e)$ is a distribution of positive mass, bounded in total amount and lying on the set E, then*

$$\int_w df(e_P) \int_w \frac{1}{MQ} dg(e_Q, P) = \int_w \frac{1}{MQ} d\left[\int_w g(e_Q, P)df(e_P)\right],$$

(or both are $+\infty$), where $\int_w g(e, P)df(e_P)$ is a bounded distribution of positive mass lying on the set F.

If

$$h^N(M, Q) = \frac{1}{MQ}, \text{ if } \frac{1}{MQ} \leqq N,$$
$$= N, \quad \text{if } \frac{1}{MQ} > N,$$

the function $h^N(M,Q)$ is continuous in M and Q.

By Theorem 3, we have

$$\int_w df(e_P) \int_w h^N(M, Q) dg(e_Q, P)$$
$$= \int_w h^N(M, Q) d\left[\int_w g(e_Q, P) df(e_P)\right].$$

Let N become infinite; we have

$$\int_w df(e_P) \lim_{n=\infty} \int_w h^N(M, Q) dg(e_Q, P)$$
$$= \int_w \frac{1}{MQ} d\left[\int_w g(e_Q, P) df(e_P)\right].$$

The interchange of integration and passing to the limit in the left-hand member is justified because the integrand is an increasing function of N, while the definition of the integral of a lower-semi-continuous function with respect to a distribution of positive mass was used in the right-hand member. Applying this definition now to the left-hand member, we have

$$\int_w df(e_P) \int_w \frac{1}{MQ} dg(e_Q, P) = \int_w \frac{1}{MQ} d\left[\int_w g(e_Q, P) df(e_P)\right].$$

This, combined with Theorem 1, establishes the theorem.

3. *Volume Averages*. This theorem just proved enables one to exhibit the distribution of mass for the spherical volume average of a potential function in a form in which it may be evaluated; this is an illustration of the advantage sometimes gained by working with the more general situation.

In the theorems that follow, $u(Q)$ is the potential at Q of the distribution of positive mass, $f(e)$, which is bounded in total amount and lies on a bounded set E.

THEOREM 5. *The average of the potential function $u(Q)$ over a spherical volume $\Gamma(r, M)$, of radius r and center M, is a potential function of a distribution of positive mass with density $(3/(4\pi r^3))f\{\Gamma(r, Q)\}$. In symbols,*

$$
\begin{aligned}
a_u(r, M) &= \frac{3}{4\pi r^3}\int_{\Gamma(r,M)} u(Q)dQ \\
&= \frac{3}{4\pi r^3}\int_w \frac{1}{MQ} f\{\Gamma(r, Q)\}dQ.
\end{aligned} \tag{2}
$$

Let us form this spherical volume average and determine its distribution of mass. We have

$$
\begin{aligned}
a_u(r, M) &= \frac{3}{4\pi r^3}\int_{\Gamma(r,M)} u(Q)dQ \\
&= \frac{3}{4\pi r^3}\int_{\Gamma(r,M)} dQ \int_w \frac{1}{PQ} df(e_P).
\end{aligned}
$$

As the integrand is a lower-semi-continuous function, the order of integration may be interchanged, so that

$$
a_u(r, M) = \frac{3}{4\pi r^3}\int_w df(e_P) \int_{\Gamma(r,M)} \frac{1}{PQ} dQ.
$$

The inner integral is the potential at P of a sphere of unit density with radius r and center at M, which is equal to the potential at M of the same sphere with center at P. Hence,

$$
\begin{aligned}
a_u(r, M) &= \frac{3}{4\pi r^3}\int_w df(e_P) \int_{\Gamma(r,P)} \frac{1}{MQ} dQ \\
&= \frac{3}{4\pi r^3}\int_w df(e_P) \int_w \frac{1}{MQ} dg(e_Q, P),
\end{aligned}
$$

where $g(e, P) = m_3\{e\cdot\Gamma(r, P)\}$, and m_3 means the three-dimensional measure of the set indicated. The quantity $g(e, P)$ is evidently continuous in P for every measurable set e.

By means of Theorem 4, this volume average may be expressed in a form which enables the mass distribution to be evaluated,

$$(3) \qquad a_u(r, M) = \frac{3}{4\pi r^3}\int_w \frac{1}{MQ}\, d\left[\int_w m_3\{e_Q\cdot\Gamma(r, P)\}\,df(e_P)\right].$$

Consider the integral,

$$v(e) = \int_w m_3\{e\cdot\Gamma(r, P)\}\,df(e_P),$$

where e is an arbitrary bounded set measurable Borel. This function is an absolutely continuous function of e; in fact

$$v(e) \leqq m_3(e)\cdot f(E),$$

and $v(e)$ is completely additive by Theorem 1.

The integrand of $v(e)$ may be expressed as a Lebesgue integral,

$$m_3\{e\cdot\Gamma(r, P)\} = \int_{e\cdot\Gamma(r,P)} 1dR = \int_e B(R, P)dR,$$

where we may define $B(R, P)$ as 1, for $RP<r$, and 0, for $RP\geqq r$. In this way $B(R, P)$ is the limit of an increasing sequence of continuous functions of R and P, and we have

$$(4) \qquad \begin{aligned} v(e) &= \int_w df(e_P)\int_e B(R, P)dR = \int_e dR\int_w B(R, P)df(e_P) \\ &= \int_e f\{\Gamma(r, R)\}\,dR. \end{aligned}$$

Substituting this result in (3) and making use of the definition of the integral of a lower-semi-continuous function, we have

$$\begin{aligned} a_u(r, M) &= \frac{3}{4\pi r^3}\int_w \frac{1}{MQ}\, d\left[\int_{e_Q} f\{\Gamma(r, R)\}\,dR\right] \\ &= \frac{3}{4\pi r^3}\lim_{N=\infty}\int_w h^N(M, Q)\,d\left[\int_{e_Q} f\{\Gamma(r, R)\}\,dR\right]. \end{aligned}$$

As $h^N(M, Q)$ is bounded and continuous, and $v(e)$ is absolutely continuous, we may change the Stieltjes integral to a Lebesgue integral. This gives

$$a_u(r, M) = \frac{3}{4\pi r^3}\lim_{N=\infty}\int_w h^N(M, Q)f\{\Gamma(r, Q)\}\,dQ$$

$$= \frac{3}{4\pi r^3}\int_w \frac{1}{MQ} f\{\Gamma(r, Q)\} dQ,$$

and the proof is complete.

It should be pointed out that the work of this section holds for the average over any three-dimensional open set. Let $s(O)$ be an open set (therefore of positive spatial measure), $s(M)$ the set obtained by displacement of $s(O)$ as a rigid body, without rotation, so that O falls on M, and let $s'(Q)$ be the reflection of the set $s(M)$ through the midpoint of the line MQ.

THEOREM 6. *The average of the potential function $u(Q)$ over the set $s(M)$ is a potential function of a distribution of positive mass with density* $(1/[m_3 s(M)])f\{s'(Q)\}$. *In symbols,*

$$a_u\{s(M)\} = \frac{1}{m_3\{s(M)\}}\int_{s(M)} u(Q)dQ = \frac{1}{m_3\{s(M)\}} \int_w \frac{1}{MQ} f\{s'(Q)\} dQ.$$

The construction of the set $s'(P)$ gives the potential at P of the set $s(M)$, of unit density, equal to the potential at M of the set $s'(P)$, of unit density; hence all the transformations made in this section on the spherical volume average are valid for the average over the set $s(M)$.

4. *Spherical Surface Average.* By means of Theorem 4, we may also determine the mass function for the average of a potential function over a spherical surface.

THEOREM 7. *The average of the potential function $u(Q)$ over the spherical surface $C(r, M)$, of radius r and center M, is a potential function of the distribution of positive mass,*

$$\frac{1}{4\pi r^2}\int_m m_2\{e\cdot C(r, P)\} df(e_P).$$

In symbols,

$$A_u(r, M) = \frac{1}{4\pi r^2}\int_{C(r,M)} u(Q)dQ$$
$$= \frac{1}{4\pi r^2}\int_w \frac{1}{MQ}\, d\left[\int_w m_2\{e_Q\cdot C(r, P)\} df(e_P)\right].$$

Treating the spherical surface average in the same manner as we have treated the volume average, we obtain

$$\begin{aligned} A_u(r, M) &= \frac{1}{4\pi r^2}\int_{C(r,M)} u(Q)dQ = \frac{1}{4\pi r^2}\int_{C(r,M)} dQ\int_w \frac{1}{QP}df(e_P) \\ &= \frac{1}{4\pi r^2}\int_w df(e_P)\int_{C(r,M)}\frac{1}{QP}dQ \\ &= \frac{1}{4\pi r^2}\int_w df(e_P)\int_{C(r,P)}\frac{1}{QM}dQ \\ &= \frac{1}{4\pi r^2}\int_w df(e_P)\int_w \frac{1}{MQ}dg(e_Q, P), \end{aligned}$$

where $g(e, P) = m_2\{e\cdot C(r, P)\}$. For a given P, this function is additive and bounded for cells e of a three-dimensional lattice, and hence can be extended by definition uniquely to all sets spatially measurable Borel.

As $g(e, P)$ is a continuous function of P for every cell e of a net L, Theorem 4 applies, and we may thereby express this average in a form that exhibits its mass function in terms of $C(r, P)$ and $f(e)$,

$$A_u(r, M) = \frac{1}{4\pi r^2}\int_w \frac{1}{MQ}\, d\left[\int_w m_2\{e_Q\cdot C(r, P)\}df(e_P)\right]. \tag{5}$$

This result requires no restriction on $f(e)$ other than those we have already stated. However, we shall state also a special case of Theorem 7.

THEOREM 8. *If $f\{\Gamma(r, Q)\}$ is differentiable with respect to r, and, for a fixed neighborhood of the given value of r, $\partial f\{\Gamma(n, Q)\}/\partial r$ is bounded independently of the point Q, then the average of the potential function $u(Q)$ over the spherical surface $C(r, M)$ is a potential function of a distribution of positive mass with density $(1/(4\pi r^2))\partial f\{\Gamma(r, Q)\}/\partial r$. In symbols,*

$$A_u(r, M) = \frac{1}{4\pi r^2}\int_w \frac{1}{MQ}\frac{\partial f\{\Gamma(r, Q)\}}{\partial r}dQ. \tag{6}$$

We have, for a rectangular cell e,

$$m_2\{e\cdot C(r, P)\} = \lim_{i=\infty}\frac{m_3\{e\cdot[\Gamma(r_i, P) - \Gamma(r, P)]\}}{r_i - r}, \tag{7}$$

where $r < r_{i+1} < r_i$ and $\lim_{i=\infty} r_i = r$.

Using the results given in (4), we have the following equality,

$$\int_w \frac{m_3\{e\cdot[\Gamma(r_i, P) - \Gamma(r, P)]\}}{r_i - r} df(e_P) = \int_e \frac{f\{\Gamma(r_i, Q)\} - f\{\Gamma(r, Q)\}}{r_i - r} dQ.$$

The integrand of the left-hand member belongs to a sequence of measurable, uniformly bounded functions, as a function of P, whose limit exists when i becomes infinite; so we let i become infinite and interchange the order of integration and pass to the limit for the left-hand member. The same considerations hold for the integrand of the right-hand member as a function of Q. Using (13), we have

$$\int_w m_2\{e\cdot C(r, P)\} df(e_P) = \int_e \frac{\partial f\{\Gamma(r, Q)\}}{\partial r} dQ.$$

The quantity $\partial f\{\Gamma(r, Q)\}/\partial r$ is non-negative. Hence we may substitute this last equation in (5) and change the Stieltjes integral into a Lebesgue integral as we did above for the volume average. Thus we have established the theorem.

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ERRATUM

In my paper entitled *On the summability of a certain class of series of Jacobi polynomials* (this Bulletin, vol. 41 (1935), pp. 541–549), the following change should be made; it conforms with the last proofs that I had seen.

Page 544, 8th line from the bottom: read $S_{n,h}^{(k)}$ instead of $S_{n,k}^{(k)}$.

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