SOME THEOREMS ON TENSOR DIFFERENTIAL INVARIANTS

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1. *Introduction.* In the theory of algebraic invariants there is a theorem which states that if an absolute invariant be written as the quotient of two relatively prime polynomials, then the numerator and denominator are relative invariants.* If we consider absolute scalar differential invariants of a metric (or affine) space, then it is possible to prove a similar theorem regarding them. In the course of the proof we give a new proof of the fact that in a relation of the form (2) the ϕ must be a power of the Jacobian of the coordinate transformation. (In the algebraic theory the u_j^i are of course constants.) This proof involves the use of the differential equations satisfied by the scalar.† In this proof it is not necessary to restrict B and ϕ to be polynomials in their arguments as is done in the usual proof of the corresponding theorem in the invariant theory. It is sufficient to assume that ϕ possesses first derivatives with respect to the u_j^i and that $B(\bar{g})$ is an analytic function of ϵ in the neighborhood of $\epsilon = 0$. We also extend the theorem to the case of tensor differential invariants of the form (5).

2. *Scalar Differential Invariants.* We consider the differential invariants of a metric space V_n with a quadratic form $g_{ij}dx^idx^j$. Let

$$A\left(g_{ij};\ \frac{\partial g_{ij}}{\partial x^k};\ \cdots;\ \frac{\partial^p g_{ij}}{\partial x^k \cdots \partial x^l}\right)$$

be an absolute scalar invariant of V_n which we take to be rational in its arguments. We can then write A in terms of the g_{ij} and their extensions $g_{ij,k\cdots l}$, and we have

$$A(g_{ij};\ 0;\ g_{ij,kl};\ \cdots) = \frac{B(g_{ij};\ 0;\ g_{ij,kl};\ \cdots)}{C(g_{ij};\ 0;\ g_{ij,kl};\ \cdots)},$$

* See, for example, H. W. Turnbull, *The Theory of Determinants, Matrices, and Invariants*, p. 277.

† T. Y. Thomas and A. D. Michal, *Differential invariants of relative quadratic differential forms*, Annals of Mathematics, vol. 28 (1927), p. 679.

where B and C are polynomials in the g_{ij} and their extensions. We may assume that B and C have no common factor. Now

$$\frac{B(g_{ij};\cdots)}{C(g_{ij};\cdots)}=\frac{B(\bar{g}_{ij};\cdots)}{C(\bar{g}_{ij};\cdots)},\tag{1}$$

under an arbitrary coordinate transformation $x\rightarrow\bar{x}$, the barred g's being the g's in the $(\bar{x})$ coordinate system. It is easily shown we must have*

$$B(\bar{g}_{ij};\cdots)=\phi(u_b^a)B(g_{ij};\cdots),\tag{2}$$

$$C(\bar{g}_{ij};\cdots)=\phi(u_b^a)C(g_{ij};\cdots),\tag{3}$$

where $u_b^a=\partial x^a/\partial\bar{x}^b$ and ϕ is a polynomial in the u's. We now prove ϕ is a power of $|\partial x/\partial\bar{x}|$, thus showing B and C are relative scalars.

Write (2) in the form

$$B(\bar{g})\phi^{-1}=B(g),$$

and consider the infinitesimal transformation

$$x^i=\bar{x}^i+\epsilon\xi^i(\bar{x}).$$

We have

$$\left(\frac{d\phi^{-1}}{d\epsilon}\right)_{\epsilon=0}=-\left(\frac{\partial\phi}{\partial u_j^i}\right)_{\epsilon=0}\left(\frac{du_j^i}{d\epsilon}\right)_{\epsilon=0}=-\left(\frac{\partial\phi}{\partial u_j^i}\right)_{\epsilon=0}\frac{\partial\xi^i(x)}{\partial x^j}.$$

As ϕ is a polynomial, so also is $\partial\phi/\partial u_j^i$, and on evaluating this last expression at $\epsilon=0$, we obtain a set of constants k_i^j, so that

$$\left(\frac{d\phi^{-1}}{d\epsilon}\right)_{\epsilon=0}=-k_i^j\frac{\partial\xi^i}{\partial x^j}.$$

Proceeding then as in the paper by Thomas and Michal, page 679, we obtain the differential equations satisfied by B in the form

$$\overset{t}{X_s}(p)B=\overset{t}{k_s}B.\tag{4}$$

Now for any function G we have†

$$(\overset{t}{X_s},\overset{l}{X_m})G=\overset{t}{\delta_m}\overset{l}{X_s}G-\overset{l}{\delta_s}\overset{t}{X_m}G,$$

* H. W. Turnbull, loc. cit.

† Thomas and Michal, loc. cit., p. 663.

and, in particular, for B we would have

$$(X_s^t, X_m^l)B = \delta_m^t X_s^l B - \delta_s^l X_m^t B = B(\delta_m^l k_s^l - \delta_s^l k_m^t).$$

But also

$$(X_s^t, X_m^l)B = X_s^t(k_m^l B) - X_m^l(k_s^t B) = B(k_m^l k_s^t - k_s^t k_m^l) = 0,$$

so that

$$\delta_m^t k_s^l - \delta_s^l k_m^t = 0,$$

from which follows $k_m^l = k\delta_m^l$, where k is a constant. Substituting in (4) we find that B is a relative scalar of weight k and* $\phi = |\partial x/\partial \bar{x}|^k$. Similar results hold for C also. As stated in §1 we can prove a generalization of this result which we state as follows.

THEOREM 1. *Given a function*

$$B\left(g_{ij};\ \frac{\partial g_{ij}}{\partial x^k};\ \cdots;\ \frac{\partial^p g_{ij}}{\partial x^k \cdots \partial x^l}\right),$$

with the law of transformation

$$B\left(\bar{g}_{ij};\ \cdots;\ \frac{\partial^p \bar{g}_{ij}}{\partial \bar{x}^k \cdots \partial \bar{x}^l}\right) = \phi(u_b^a)B\left(g_{ij};\ \cdots;\ \frac{\partial^p g_{ij}}{\partial x^k \cdots \partial x^l}\right),$$

where ϕ possesses first derivatives in the u's and $B(\bar{g})$ is analytic in the neighborhood of $\epsilon = 0$. Then ϕ is a power of the Jacobian and B is a relative scalar differential invariant.

3. *Tensor Differential Invariants.* Consider the absolute tensor differential invariant with components of the form

$$T_{i\cdots j}^{a\cdots b} = \frac{U_{i\cdots j}^{a\cdots b}(g_{kl};\ g_{kl,mq};\ \cdots)}{D(g_{kl};\ \cdots)}, \tag{5}$$

where the U's and D are polynomials (with no common factor) in their arguments. Corresponding to (1) in the scalar case we have

$$\frac{U_{i\cdots j}^{a\cdots b}(g)u_k^i \cdots u_l^j}{U'^{m\cdots p}_{k\cdots l}(g,\ u)u_m^a \cdots u_p^b} = \frac{D(g)}{D'(g,\ u)} = \frac{Q(g,\ u)}{P(g,\ u)}, \tag{6}$$

* Thomas and Michal, loc. cit.

(P, Q having no common factor), where the primed U's and D represent the result of replacing the barred g's by their values in terms of the g's and u's in the expressions $U^{\cdots}_{\cdots}(\bar{g})$ and $D(\bar{g})$. From (6) we see that Q is a factor of D and of $U^{a\cdots b}_{i\cdots j}u^i_k \cdots u^j_l$, and hence must be a function $Q'(g)$, so that

$$U^{a\cdots b}_{i\cdots j}(g)u^i_k \cdots u^j_l = Q'(g)V^{a\cdots b}_{k\cdots l}(g, u), \tag{7}$$

$$D(g) = Q'(g)R(g). \tag{8}$$

In (7) put $u^i_j = \delta^i_j$; then

$$U^{a\cdots b}_{k\cdots l}(g) = Q'(g)V^{a\cdots b}_{k\cdots l}(g, \delta). \tag{9}$$

Hence $Q'(g) = \text{const.}$, since D and the U's have no common factor. Since $D(g)$ and $D'(g, u)$ are of the same degree in the g's, it then follows from (6) that $P(g, u) = P'(u)$, so that

$$D(\bar{g}) = \phi(u^i_j)D(g).$$

Hence as proved in the previous section for B, *we have shown that D is a relative scalar of weight k, and therefore $U^{a\cdots b}_{i\cdots j}$ are the components of a relative tensor of weight k.* We can also prove the following theorem.

THEOREM 2. *If the set of quantities*

$$T^{i\cdots j}_{a\cdots b}\left(g_{kl}; \cdots ; \frac{\partial^p g_{kl}}{\partial x^m \cdots \partial x^r}\right)$$

have the transformation law

$$T^{s\cdots t}_{v\cdots w}(\bar{g}_{kl}; \cdots)u^i_s \cdots u^j_t = \phi(u^d_e)T^{i\cdots j}_{a\cdots b}(g_{kl}; \cdots)u^a_v \cdots u^b_w,$$

then ϕ is a power of $|\partial x/\partial \bar{x}|$ and the T's are components of a relative tensor invariant, it being assumed that ϕ possesses first derivatives in the u's, and $T(\bar{g})$ are analytic in the neighborhood of $\epsilon = 0$.

The proof is similar to that used for B of the previous section. Similar results to those obtained for metric scalar and tensor differential invariants hold for affine invariants.

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