in the case of algebraic closure there is a stronger result of Hollkott (Hamburg): the axiom of choice is sufficient for the existence and uniqueness of algebraic closure.

An essential simplification is made possible for Baer's† theory of the degree of algebraic extensions. I plan to show elsewhere how the generalized continuum hypothesis may be avoided.

YALE UNIVERSITY

CONCERNING TWO INTERNAL PROPERTIES OF PLANE CONTINUA‡

BY R. E. BASYE

Theorem 1 below was suggested to me by R. L. Moore. Theorem 2 is an extension of Kuratowski's result§ that if three compact plane continua have a point in common and their sum separates a point A from a point B in the plane, then there exists a pair of these continua whose sum separates A from B in the plane. Another extension of this result along combinatorial lines has been given‖ by Čech.

THEOREM 1. *Let H and K be two mutually exclusive and closed subsets of a compact continuum M which lies in the plane. If for each pair of points A and B in H and K, respectively, there exists a finite collection Γ_{AB} of continua in M such that Γ_{AB}^* separates A from B in M, then there exists a finite collection Γ of continua in M such that Γ^* separates H from K in M.*

Let $\epsilon_1, \epsilon_2, \cdots$, be a sequence of positive numbers converging monotonically to zero, with ϵ_1 less than half the distance from H to K. For each i let D_H^i be a domain containing H such that (1) the boundary β_H^i of D_H^i is the sum of a finite number of mutually exclusive simple closed curves, and (2) each point of

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† *Eine Anwendung der Kontinuumhypothese in der Algebra.* Journal für Mathematik, vol. 162.

‡ Presented to the Society, April 6, 1935, under a somewhat different title.

§ Kuratowski, *Théorème sur trois continus,* Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 77–80.

‖ E. Čech, *Trois théorèmes sur l'homologie,* Publications de la Faculté des Sciences de L'Université Masaryk, No. 144, 1931, pp. 1–21.

D_H^i lies at a distance less than ϵ_i from H. For each i let D_K^i be a similar domain containing K and having boundary β_K^i. We shall show that there exists a value of i such that the number of components of $M-M\cdot(D_H^i+D_K^i)$ which intersect both β_H^i and β_K^i is finite.

Suppose there are infinitely many such components for each value of i. Consider for each i an infinite sequence $C_1^i, C_2^i, \cdots$, of components of $M-M\cdot(\overline{D_H^i}+\overline{D_K^i})$, no two of which lie in the same component of $M-M\cdot(D_H^i+D_K^i)$, and such that both β_H^i and β_K^i contain a limit point of $C_j^i\,(j=1, 2, \cdots)$. There exists an infinite subsequence $\alpha^i=E_1^i, E_2^i, \cdots$ of the sequence $\overline{C_1^i}, \overline{C_2^i}, \cdots$, and two components j_H^i, j_K^i of β_H^i, β_K^i, respectively, such that each element of α^i intersects both j_H^i and j_K^i.

No component of β_H^i or β_K^i distinct from j_H^i and j_K^i can intersect more than two elements of α^i. For suppose j is such a component, intersecting, say, the three elements E_1^i, E_2^i, E_3^i of α^i. The continua E_1^i, E_2^i, E_3^i are mutually exclusive, intersect j, and are not disconnected by j. It follows that no two points of $j_H^i\cdot E_n^i$, $(n=1, 2, 3)$, are separated from each other in j_H^i by $\sum_{r=1}^3 E_r^i - E_n^i$. Hence there exist three arcs b_1, b_2, b_3 such that (1) $b_1+b_2+b_3=j_H^i$, and (2) b_n, $(n=1, 2, 3)$, contains $j_H^i\cdot E_n^i$ but no point of $\sum_{r=1}^3 E_r^i - E_n^i$. Denote by F_n^i, $(n=1, 2, 3)$, the continuum $E_n^i+b_n+j$. Let δ_H^i and δ_K^i denote the complementary domains of j_H^i and j_K^i which do not contain j_K^i and j_H^i, respectively. The three continua F_n^i, $(n=1, 2, 3)$, have a point in common and their sum separates a point X of δ_H^i from a point Y of δ_K^i in the plane. Hence, by a theorem of Kuratowski,† there exist two of these continua, say F_1^i and F_2^i, such that $F_1^i+F_2^i$ separates X from Y in the plane. But $\delta_H^i+(E_3^i-j\cdot E_3^i)+\delta_K^i$ is a connected set which contains X and Y but no point of $F_1^i+F_2^i$.

Consequently, the number of components of $\beta_H^i+\beta_K^i$ being finite, there exists an integer r such that each element of the sequence $\gamma^i=E_r^i, E_{r+1}^i, \cdots$, intersects both components j_H^i, j_K^i without meeting any other component of $\beta_H^i+\beta_K^i$. The intersection of each element E_n^i of γ^i with j_H^i is contained in an arc (or point) of j_H^i which intersects no other element of γ^i. Denote by a_n^i the minimal such arc (or point), and by b_n^i a

† Loc. cit.

similar arc (or point) containing the intersection of E_n^i with j_K^i. The sequence $\{a_n^i+b_n^i\}$, $(n=r, r+1, \cdots)$, contains a subsequence which has a sequential limiting set; let P_i and Q_i denote points of this limiting set which lie on j_H^i and j_K^i respectively. The sequence $\{P_i+Q_i\}$ contains a subsequence which has a sequential limiting set; let P and Q denote points of this limiting set which lie in H and K, respectively.

By hypothesis there exists a finite collection Γ_{PQ} of continua in M such that Γ_{PQ}^* separates P from Q in M. Hence $M-\Gamma_{PQ}^* = N_P+N_Q$, where N_P and N_Q are mutually separated sets containing P and Q, respectively. Let R_P and R_Q be regions containing P and Q, respectively, such that $R_P\cdot(N_Q+\Gamma_{PQ}^*)=R_Q\cdot(N_P+\Gamma_{PQ}^*)=0$. Let i be chosen so that P_i and Q_i lie in R_P and R_Q, respectively. Then there exists an infinite subsequence $\epsilon^i=E_{n,1}^i, E_{n,2}^i, \cdots$ of γ^i such that $a_{n,t}^i$ and $b_{n,t}^i$, $(t=1, 2, \cdots)$, are subsets of R_P and R_Q, respectively. Every element of ϵ^i intersects some element of Γ_{PQ}. Therefore some element W of Γ_{PQ} intersects at least two elements ϵ^i, hence contains a point of $\beta_H^i+\beta_K^i$. Hence, if $E_{n,s}^i$ denotes an element of ϵ^i which intersects W, then W contains a point of $a_{n,s}^i+b_{n,s}^i$. But $a_{n,s}^i+b_{n,s}^i$ is a subset of R_P+R_Q, which contains no point of Γ_{PQ}^*.

It follows that there exists a value of i such that the collection Γ of components of $M-M\cdot(D_H^i+D_K^i)$ which intersect both β_H^i and β_K^i is finite. Denote by U_H the sum of those components of $M-M\cdot(D_H^i+D_K^i)$ which intersect β_H^i but not β_K^i, and by U_K the sum of those components which intersect β_K^i but not β_H^i. It can be seen that $M\cdot D_H^i+U_H$ and $M\cdot D_K^i+U_K$ are mutually separated sets containing H and K, respectively. Hence Γ^* separates H from K in M.

R. L. Moore has proved† the closely related result whose statement is identical with that of Theorem 1 except that each Γ_{AB} and Γ are taken as finite sets. The argument is valid in any space satisfying his axioms 0 and 1. An example showing that Theorem 1 is not true for three-dimensional continua has been given by W. T. Reid.‡

The conclusion of Theorem 1 will also hold if Γ_{AB}^* merely

† *Foundations of Point Set Theory*, Colloquium Publications of this Society, vol. 13, p. 140, Theorem 57.

‡ See this issue of this Bulletin, pp. 683–684.

weakly disconnects† A from B in M, provided M is locally connected at each point of $H+K$. The argument is for the most part similar.

Again, if we require only that Γ_{AB} be a countable collection of continua in M such that Γ_{AB}^* is closed and separates A from B in M, then there exists a countable collection Γ of continua in M such that Γ^* is closed and separates H from K in M.

THEOREM 2. *Let M be any subcontinuum of a plane or a sphere and let Z be any subset (possibly vacuous) of M. If G is a finite collection of connected subsets of M such that* (1) *there exist r points (r finite) whose sum L intersects each element of G, and* (2) *G^*+Z weakly disconnects a point A from a point B in M, then there exists a subcollection H of G, containing not more than $2r$ elements, such that H^*+Z weakly disconnects A from B in M.*

The case where M is a subset of a plane is a consequence of the case where M is a subset of a sphere. Suppose, then, that M is a subcontinuum of a sphere S. We shall discuss in order the three cases that may arise.

CASE 1. Suppose $r=1$. Assume the theorem false. Let $g_1, \cdots, g_n$ denote the elements of G and let O be a point common to these elements. There exist continua C_{ij}, $(i, j=1, \cdots, n)$, such that C_{ij} contains $A+B$ and has no point in common with g_i+g_j+Z. Let D_k, $(k=1, \cdots, n)$, be a connected domain which contains g_k and contains no point of any C_{kj}, $(j=1, \cdots, n)$. Let D_0 be a domain which contains Z but contains no point of any C_{ij}, $(i, j=1, \cdots, n)$. Let $\Delta=\sum_{i=0}^{n} D_i$. The domain $(S-M)+\Delta$ weakly disconnects, hence separates, A from B in S. Hence there exists a component Q of $(S-M)+\Delta$ which separates A from B in S, and Q must contain the connected set $\sum_{i=1}^{n} D_i$.

If Q_i, $(i=1, \cdots, n)$, denotes that component of $(S-M)+D_0+D_i$ which contains g_i, then $Q=Q_1+\cdots+Q_n$. For suppose the contrary and let R denote the set $Q-(Q_1+\cdots+Q_n)$. Since Q_1+R is not connected, R contains a subset R_1 such that R_1 and $(Q_1+R)-R_1$ are mutually separated sets. In general, if R_i has been defined, R_{i+1}, $(i=1, \cdots, n-1)$, will

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† A set L weakly disconnects a set A from a set B in a connected set M if it contains a point of every connected and relatively closed subset of M which intersects both A and B.

denote a subset of R_i such that R_{i+1} and $(Q_{i+1}+R_i)-R_{i+1}$ are mutually separated sets. It follows that R_n and $Q-R_n$ are mutually separated sets, contrary to the connectedness of Q.

Let F_{it}, $(i=1, \cdots, n;\ t=1, 2, \cdots)$, be a subcontinuum of Q_i which contains O and every point of Q_i whose distance from the boundary of Q_i is greater than $1/t$. Let $F_t=F_{1t}+\cdots+F_{nt}$, $(t=1, 2, \cdots)$. There exists a value w of t such that F_w separates A from B in S. For on the contrary supposition there exists a sequence of continua $N_1, N_2, \cdots$, such that N_t, $(t=1, 2, \cdots)$, contains $A+B$ and contains no point of F_t. The limiting set of this sequence is a continuum which contains $A+B$ but no point of $Q_1+\cdots+Q_n$. But this is impossible. Consequently† there exist two of the continua F_{iw}, say F_{1w} and F_{2w}, whose sum separates A from B in S. Hence Q_1+Q_2 separates A from B in S. But the continuum C_{12} contains $A+B$ and has no point in common with Q_1+Q_2.

CASE 2. Suppose that $r>1$ and G^* is connected. There exists a subcollection K of G having not more than $2r-2$ elements and such that K^* is a connected set which contains L. Denote by G' the collection of all elements each of which is the sum of K^* and an element of G. The elements of G' are connected sets having K^* in common; hence, by Case 1, there exist two elements g_1' and g_2' of G' such that $g_1'+g_2'+Z$ weakly disconnects A from B in M. The set $g_1'+g_2'$ is the sum of $2r$ or less elements of G.

CASE 3. Suppose merely that $r>1$. Let H be a subcollection of G which is irreducible with respect to the property that H^*+Z weakly disconnects A from B in M. Let $C_1, \cdots, C_q$ denote the components of H^* and let r_i, $(i=1, \cdots, q)$, be the number of points common to L and C_i. By Case 2 there exist $2r_i$ or less elements of H lying in C_i, $(i=1, \cdots, q)$, such that, if T_i denotes their sum, $T_i+[Z+(H^*-C_i)]$ weakly disconnects A from B in M. Hence C_i cannot contain more than $2r_i$ elements of H. Therefore, since $r\geqq r_1+\cdots+r_q$, H^* cannot contain more than $2r$ elements of G.

THE UNIVERSITY OF TEXAS

† C. Kuratowski, loc. cit.