BESICOVITCH ON ALMOST PERIODIC FUNCTIONS

Almost Periodic Functions. By A. S. Besicovitch. Cambridge, University Press, 1932. xiii+180 pp.

The seven years which elapsed between the time of creation of the theory of almost periodic functions by H. Bohr in 1925 and publication of Besicovitch's book, were years of remarkable development. The notion and properties of almost periodic functions, either in their initial, or in generalized form turned out to be of great importance in various fields of analysis, function theory, topology, and applied mathematics. The necessity of a monograph giving a concise and systematic exposition of the fundamentals of the theory of almost periodic functions was becoming more and more obvious. The task of writing such a monograph was an arduous one, but it is not astonishing that the author has completely succeeded in solving this difficult problem. The present monograph is but one of the numerous important contributions of the author to the theory of almost periodic functions for which he was awarded an Adams prize of the University of Cambridge in 1931. According to his own statement the author did not intend to give an encyclopaedic account of the theory; on the other hand, he was extremely painstaking in giving a clear and complete picture of the logical structure of the theory.

The book contains three chapters and an appendix between Chapters 2 and 3. Chapter 1, *Uniformly almost periodic functions* (66 pp.), is devoted to a discussion of continuous almost periodic functions as introduced initially by H. Bohr. The central point of the chapter is of course the proof of H. Bohr's "Fundamental Theorem," $\sum|A_n|^2=M\{|f(x)|^2\}$. The initial proof of this theorem as given by H. Bohr was very complicated. Several other proofs were given later (Weyl, Wiener, de la Vallée-Poussin). The author gives the simplest of these proofs, which is due to de la Vallée-Poussin, and is based on a combination of ideas of Weyl and Bohr. Considerable attention is given also to the problem of summation of the Fourier series of an almost periodic function by means of sequences of partial sums, and also by means of the associated Fejér-Bochner trigonometric polynomials. A special paragraph is devoted to a discussion of arithmetical nature of translation numbers of an almost periodic function. The author considers the set $\overline{E}_\epsilon=\overline{E}(\epsilon; f(x))$ of all integers which are ϵ-translation numbers of $f(x)$, introduces the notion of almost periodicity of such sets, and proves that $\overline{E}_\epsilon$ is almost periodic for almost all values of ϵ. This theorem, due to the author jointly with H. Bohr, is interesting in itself and also plays an important role in the subsequent development of the theory of generalized almost periodic functions. The chapter closes with a rapid sketch of uniformly almost periodic functions of two independent variables.

Chapter 2, *Generalization of almost periodic functions* (56 pp.), treats of various generalizations of the notion of an almost periodic function. These generalizations, in the increasing order of their generality, are as follows: (S^p=Stepanoff) a.p.$\subset$ (W^p=Weyl) a.p.$\subset$ (B^p=Besicovitch-Bohr) a.p., $p\geqq 1$. They all include as the most special case the (U=uniformly) a.p. functions of Bohr. While (S^p) and (W^p) a.p. functions represent a "natural" generalization of the (U) a.p. functions, which are obtained by merely modifying the metric of the underlying function space, the notion of the (B^p) a.p. involves a more radical and farther distant modification. It is significant that the recent generalizations of the theory of almost periodic functions of v. Neumann and Bochner, while being fully capable of including the theory of the (S^p) and (W^p) a.p. functions, appear to be less effective for the theory of the (B^p) a.p. functions. Let $\mathfrak{A}$ be the set of all finite trigonometric polynomials and let C_G denote the operation of taking the closure in a space $G = U, S^p, W^p$, or B^p. The central point of the theory consists in showing that for each choice of G, $(G) = C_G(\mathfrak{A})$. The relation $C_G(\mathfrak{A}) = C_G\{C_U(\mathfrak{A})\}$ turns out to be of fundamental importance. Fejér-Bochner trigonometric polynomials are shown to be as efficient for summation of the Fourier series of a function of any (G) as they were in the special case of $G = U$. The climax of the whole theory is reached at the end of the chapter when the author states and proves the Riesz-Fischer Theorem: *To each formal series* $\sum A_n e^{i\lambda_n x}$ *with* $\sum |A_n|^2 < \infty$ *there corresponds a* (B^2) *a.p. function which has this series as its Fourier series.* This theorem does not hold true for the (U), (S^2) or (W^2) a.p. functions, although the converse theorem (Parseval's Theorem) holds for each admissible G. In the following appendix (17 pp.) the author discusses a modified notion of the $(\overline{B})$ a.p. functions which is not so general as that of (B) but possesses some advantages of its own.

The last chapter, Chapter 3, *Analytic almost periodic functions* (43 pp.), is devoted to an exposition of the theory of analytic functions almost periodic in a strip, due largely to H. Bohr. The author does not give much space to the discussion of the problem of distribution of values of such functions. On the other hand he adds several important contributions of his own, among which we shall mention only two. (i) An extension of Picard's Theorem to a general class of functions uniformly a.p. in a half-plane. (ii) A proof of the remarkable theorem to the effect that if a function is uniformly a.p. in an open strip (α, β) and can be continued analytically into a larger open strip (α_1, β_1), then, under certain restrictions of a very general form, the extended function will still be (B^2) a.p. in (α_1, β_1).

The exposition, although condensed, is extremely clear and suggestive throughout the whole book. The author has succeeded in his endeavor to make his book self contained by eliminating as many references to other sources as possible. Thus he inserted proofs of various auxiliary propositions necessary for the general discussion, which will be of greatest help to the reader. Misprints are not entirely non-existent in the book, but their number is very small. It is curious that the name of Bohl, who introduced the notion of quasi-periodic functions is confused with that of Bohr, and that the name of F. Riesz is consistently spelled as F. Riesc.

Taken as a whole, the *Almost periodic functions* of Besicovitch represents a notable contribution whose study is indispensable to every one working in the field of almost periodic functions, and which is destined to become a starting point of many an important investigation in this fascinating theory.

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