A NOTE ON THE EQUILIBRIUM POINT OF THE GREEN'S FUNCTION FOR AN ANNULUS

DEBORAH M. HICKEY

1. *Introduction.* In a previous paper* the motion of the equilibrium point of the Green's function for a plane annular region was studied as the pole was shifted along a radius in the neighborhood of the geometric mean circle C_0.† The expression for dr/dr_0 on C_0, r being the distance of the equilibrium point from the center of the circles, r_0 that of the pole, is $-F_{r_0}/F_r$, where

$$F_{r_0} = \frac{\partial F}{\partial r_0} = -\frac{2}{R}\left[\frac{1}{2\log R} - \frac{1}{8} + \sum_{m=1}^{\infty}\frac{(-1)^m m}{R^m - 1}\right],$$

$$F_r = \frac{\partial F}{\partial r} = -\frac{2}{R}\left[\frac{1}{8} + \sum_{m=1}^{\infty}\frac{(-1)^m m}{R^m + 1}\right].$$

In these formulas 1 and R are the radii of the inner and outer circular boundaries of the region. It was shown by an application of a theorem of Schlömilch‡ that F_{r_0} does not vanish on C_0.

In this article this result and others are obtained by a method which seems better adapted to the problem.§

It is noticed that the function

$$f(z) = \frac{\pi}{\sin \pi z}\,\frac{z}{e^{az} - 1}, \qquad a = \log R,$$

---

* D. M. Hickey, *The equilibrium point of Green's function for an annular region*, Annals of Mathematics, vol. 30 (1929), pp. 373–383.

† The Green's function for this region may be written in the form

$$g(M, M_0) = \log\frac{1}{MM_0} + \frac{1}{\log R}[\log R \log r_0 - \log r \log r_0/R]$$

$$- \sum_{m=1}^{\infty}\frac{1}{m}\,\frac{\cos m(\theta - \theta_0)}{R^{2m} - 1}\left\{r^m[r_0^m - r_0^{-m}] + r^{-m}\left[\left(\frac{R^2}{r_0}\right)^m - r_0^m\right]\right\}.$$

We take $F(r, r_0) = \partial g/\partial r$ for $r = r_0 = R^{1/2}$ and $\theta - \theta_0 = \pi$.

‡ *Über einige unendliche Reihen*, Zeitschrift für Mathematik und Physik, vol. 23 (1878), p. 132.

§ The suggestion that the method of contour integration and the theory of residues might prove useful was given by A. J. Maria.

where R is a real number greater than 1, has the sum of residues $\sum_{m=1}^{n}(-1)^m m/(R^m-1)$ within a suitably selected contour containing as singularities only the poles $1, 2, \cdots, n$ of $f(z)$. It turns out by an integration around a contour and certain limiting processes that an expression for F_{r_0} on C_0 is found which is a series of positive terms. The same method applied to a suitably chosen function $f(z)$ yields for the value of F_r a series of negative terms. These values show that dr/dr_0 is positive on C_0. By means of the preceding results it is proved that d^2r/dr_0^2 is negative on C_0.*

2. *The Evaluation of F_{r_0} and F_r on C_0 by Contour Integration.* The contour C_n chosen for the evaluation of F_{r_0} consists of the lines $x_n=n+1/2$, $y_n=\pm(2n+1)\pi/a$, semi-circular arcs that lie to the right of the imaginary axis of positive radius $\rho<\pi/(2a)$ and <1 and centers $\pm 2m\pi i/a$, $(m=0, 1, \cdots, n)$, and the straight line segments of the imaginary axis exterior to these arcs included between the upper and lower y_n lines. The function

$$f(z) = \frac{\pi}{\sin \pi z}\cdot\frac{z}{e^{az}-1}$$

is analytic inside and on C_n except at the poles $z=1, 2, \cdots, n$ of $\pi/\sin\pi z$. Hence the value of the integral $(1/2\pi i)\int_{C_n} f(z)dz$, where the contour C_n is traced in the counter-clockwise direction, gives $\sum_{m=1}^{n}(-1)^m m/(e^{am}-1)$, the sum of the residues of $f(z)$ inside C_n.

Let L_n be the straight line segments on the imaginary axis, K_n the semi-circular arcs, and S_n the remaining part of C_n. Over L_n the integral can easily be put into the form

$$-\frac{1}{2}\sum_{m=0}^{n-1}\int_{(2(m+1)\pi/a)-\rho}^{(2m\pi/a)+\rho}\frac{ydy}{\sinh \pi y}-\frac{1}{2}\int_{(2n+1)\pi/a}^{(2n\pi/a)+\rho}\frac{ydy}{\sinh \pi y}.$$

For the evaluation of the integral over the arc of K_n with center at $2m\pi i/a$, $(m\neq 0)$, a power series development of $f(z)$ about this point is used. Evaluated, the integral gives

$$-\frac{\pi^2}{a^2}\frac{m}{\sinh\,(2m\pi^2/a)}+P_m(\rho),$$

* It is evident that the corresponding results hold for any annulus.

where $P_m(\rho)$ is a power series in ρ with constant term zero. Over the arc with center at the origin the value is found to be $(-1/(2a))+P_0(\rho)$. Thus integration around C_n gives

$$
\begin{aligned}
\sum_{m=1}^{n} \frac{(-1)^m m}{e^{am}-1} &= \frac{1}{2}\sum_{m=0}^{n-1}\int_{(2m\pi/a)+\rho}^{(2(m+1)\pi/a)-\rho} \frac{y\,dy}{\sinh \pi y} \\
(1) \qquad &+ \frac{1}{2}\int_{(2n\pi/a)+\rho}^{((2n+1)\pi/a)-\rho} \frac{y\,dy}{\sinh \pi y} - \frac{1}{2a} \\
&- \frac{2\pi^2}{a^2}\sum_{m=1}^{n} \frac{m}{\sinh (2m\pi^2/a)} + \sum_{m=-n}^{n} P_m(\rho) + \frac{1}{2\pi i}\int_{S_n} f(z)dz.
\end{aligned}
$$

The value of $(1/2\pi i)\int_{C_n} f(z)dz$ is clearly the same for any positive ρ less than both $\pi/(2a)$ and 1. Letting ρ approach zero in (1), we obtain

$$
\begin{aligned}
\sum_{m=1}^{n} \frac{(-1)^m m}{e^{am}-1} &= \frac{1}{2}\int_{0}^{(2n+1)\pi/a} \frac{y\,dy}{\sinh \pi y} \\
(2) \qquad &- \frac{2\pi^2}{a^2}\sum_{m=1}^{n} \frac{m}{\sinh (2m\pi^2/a)} - \frac{1}{2a} + \frac{1}{2\pi i}\int_{S_n} f(z)dz.
\end{aligned}
$$

Now let n become infinite. The left member of (2) has as limit the convergent series $\sum_{m=1}^{\infty}(-1)^m m/(e^{am}-1)$. The first term on the right approaches the definite integral

$$\frac{1}{2}\int_{0}^{\infty} \frac{y\,dy}{\sinh \pi y},$$

which is known to have the value 1/8. The series approaches

$$-\frac{2\pi^2}{a^2}\sum_{m=1}^{\infty} \frac{m}{\sinh (2m\pi^2/a)}.$$

The integral over S_n has the limit zero.

To prove this last statement consider the modulus of $\int_{S_n} f(z)dz$. It can be shown* that over the entire curve S_n, $|1/\sin \pi z|$ and $|1/(e^{az}-1)|$ are bounded independently of n.

---

* That $|1/\sin \pi z|$ is bounded on S_n independently of n is proved essentially by Lindelöf in *Théorie des Résidus*, 1905, p. 32, footnote. The statement for $|1/(e^{az}-1)|$ can be proved in the same manner.

Let M be the greater of these two bounds. Moreover, on the upper and lower y_n lines

$$\left|\frac{1}{\sin \pi z}\right| < \frac{1}{\sinh((2_{n+1})\pi^2)/a)},$$

and on the right-hand boundary $z=(n+1/2)+iy$,

$$\left|\frac{1}{e^{az}-1}\right| < \frac{1}{e^{a(n+1/2)}-1}.$$

It then follows easily that

$$\left|\int_{S_n} f(z)dz\right| < \frac{Mk(2n+1)^2}{2a}\left[\frac{1}{2\sinh((2n+1)\pi^2/a)} + \frac{1}{e^{a(n+1/2)}-1}\right],$$

where k is a constant independent of n. This is sufficient to prove the statement.

In the limit for n infinite, (2) gives

$$\text{(3)} \qquad \frac{1}{2a} - \frac{1}{8} + \sum_{m=1}^{\infty} \frac{(-1)^m m}{e^{am}-1} = -\frac{2\pi^2}{a^2}\sum_{m=1}^{\infty}\frac{m}{\sinh(2m\pi^2/a)}.$$

The left side of (3), where a is replaced by $\log R$, multiplied by $-2/R$, is F_{r_0}. Thus F_{r_0} is *positive*.

For the evaluation of F_r let

$$\frac{\pi}{\sin \pi z}\,\frac{z}{e^{az}+1}$$

be chosen for $f(z)$. Let the contour of integration C_n consist of the lines $x_n=n+1/2$, $y_n=\pm 2n\pi i/a$, semi-circular arcs to the right of the imaginary axis of radius $\rho<\pi/(2a)$ and with their centers at the points $\pm(2m+1)\pi i/a$, $(m=0, 1, \cdots, n)$, and the portions of the imaginary axis exterior to these arcs between the upper and lower y_n lines.

Applied to this function over the chosen contour, the method used above yields easily the result

$$\text{(4)} \qquad \frac{1}{8} + \sum_{m=1}^{\infty}\frac{(-1)^m m}{e^{am}+1} = \frac{\pi^2}{a^2}\sum_{m=0}^{\infty}\frac{2m+1}{\sinh((2m+1)\pi^2/a)}.$$

By use of (4) with $\log R = a$, F_r can be written as

$$-\frac{2\pi^2}{R(\log R)^2}\sum_{m=1}^{\infty}\frac{(2m+1)}{\sinh\left(\frac{(2m+1)\pi^2}{\log R}\right)},$$

a series of *negative terms*. With these values for F_{r_0} and F_r, we conclude that $dr/dr_0 = -F_{r_0}/F_r$ on C_0 is positive.

3. *The Sign of d^2r/dr_0^2 on C_0.* From $dr/dr_0 = -F_{r_0}/F_r$, we calculate the second derivative

$$\frac{d^2r}{dr_0^2} = F_r^{-3}\left[2F_{r_0}F_rF_{r_0r} - F_{r_0}^2F_{rr} - F_r^2F_{r_0r_0}\right]. \tag{5}$$

From the general expressions for F_{r_0} and F_r in terms of r, r_0, and R, the following relations on C_0 are found to hold

$$F_{r_0r_0} = -R^{-1/2}F_{r_0}, \quad F_{rr} = -3R^{-1/2}F_r, \quad F_{r_0r} = -R^{-1/2}F_{r_0}.$$

A substitution of these values in (5) gives

$$\frac{d^2r}{dr_0^2} = R^{-1/2}F_{r_0}F_r^{-2}\left[F_{r_0} + F_r\right]. \tag{6}$$

Since F_{r_0} and F_r^2 are positive, the sign of d^2r/dr_0^2 on C_0 is that of $F_{r_0}+F_r$. From the results of the preceding section we have

$$F_{r_0} + F_r = \frac{2\pi^2}{R(\log R)^2}\sum_{m=1}^{\infty}\frac{(-1)^m m}{\sinh\left(\frac{m\pi^2}{\log R}\right)}.$$

This alternating series converges to a negative sum since its terms are in absolute value strictly decreasing to zero. This shows that d^2r/dr_0^2 is negative on C_0.

MISSISSIPPI DELTA STATE TEACHERS COLLEGE