NOTE ON NON-ANALYTIC FUNCTIONS*

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In his retiring presidential address before the American Mathematical Society (December, 1931), E. R. Hedrick gave a resumé of the theory of non-analytic functions of a complex variable.† We propose in the present note to establish, with regard to these functions, some elementary properties that do not appear in the Hedrick report. (In the course of the work, however, some properties considered by Hedrick will present themselves.)

Let

$$(1) \qquad w = u + iv = f(z), \qquad z = x + iy,$$

where u, v have continuous first partial derivatives in x and y, in some plane region $\mathcal{R}$. Equation (1) can be expressed as a point transformation in the plane:

$$(2) \qquad T\colon \ u = u(x, y), \qquad v = v(x, y).$$

If f is non-analytic in $\mathcal{R}$ (as we shall assume throughout), then T is not a *directly conformal* transformation; that is, the magnitude and sense of angles are not both preserved under T. Let m_1, m_2 be (the slopes of) any two directions at a point $z=x+iy$. In general they will transform into directions not forming the same angle (magnitude and sense both considered). In fact, the following theorem is easily verified by elementary methods.

THEOREM 1. *A necessary and sufficient condition that two distinct directions m_1, m_2 at a point $z=x+iy$ transform conformally‡ under T is that m_1 and m_2 satisfy the relation*

$$(3) \qquad (G - J)m_1m_2 + F(m_1 + m_2) + (E - J) = 0.$$

Here

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* Presented to the Society, April 18, 1930.

† *Non-analytic functions of a complex variable*, this Bulletin, vol. 39 (1933), pp. 75–96.

‡ By *conformal* we shall mean *directly conformal.*

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(4) \quad
\begin{aligned}
E &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2, & F &= \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}, \\
G &= \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2, & J &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}.
\end{aligned}
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Among the quantities E, F, G, J there are some easily established identities, of which the following are of use to us:

$$(5) \qquad J^2 = EG - F^2,$$

$$(6) \qquad (J - E)\frac{\partial v}{\partial y} + F\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (J - G)\frac{\partial u}{\partial x} = 0,$$

$$(7) \qquad (J - E)\frac{\partial u}{\partial y} + F\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - (J - G)\frac{\partial v}{\partial x} = 0.$$

A point (x, y) may be such that all angles at the point transform conformally. A (necessary and sufficient) condition for this is that the Cauchy-Riemann equations be satisfied at the point:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

We shall consider only such points as *do not have the above property.*

COROLLARY. *With each direction m_1 at $z = x + iy$, there is associated a unique direction m_2, which we term the dual direction, such that the angle formed by m_1, m_2 transforms conformally. If m_2 is the dual of m_1, then m_1 is the dual of m_2.*

Of the pairs of dual directions m_1, m_2 at a point, three have special significance: *self-dual* or *contact* directions (of which there are two) and *ortho* directions.* These are defined as dual directions which are, respectively, coincident and orthogonal; that is, for which, respectively, $m_2 = m_1$, and $m_2 = -1/m_1$. Referring to (3), we have (on setting $m = dy/dx$) the following theorem.

THEOREM 2. *The contact directions and the ortho directions are defined respectively by*

* The ortho directions are termed *principal directions* by Hedrick, loc. cit., p. 78, and were first found by Tissot.

$$(G - J)dy^2 + 2Fdydx + (E - J)dx^2 = 0, \tag{8}$$

$$Fdy^2 + (E - G)dydx - Fdx^2 = 0. \tag{9}$$

As (x, y) varies, (8) and (9) define two systems of curves, which we may term the system of *contact curves* and the system* of *characteristic curves*. The discriminant of (9) is non-negative.

COROLLARY. *The characteristic curves form a real orthogonal system.*

On the other hand, the contact curves need not be real. In fact, the discriminant of (8) is

$$4J(E + G - 2J) = 4J\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right].$$

COROLLARY. *The contact curves are real at (and only at) those points (x, y) where $J \geqq 0$.*

The function $f(z)$ being non-analytic, a unique derivative does not exist. Rather, *in each direction m* at a point, there is a unique derivative. One of the forms in which it can be written is

$$\begin{aligned} \frac{df}{dz_m} = \frac{1}{(1 + m^2)}\Bigg[&\left\{\frac{\partial u}{\partial x} + m\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + m^2\frac{\partial v}{\partial y}\right\} \\ &+ i\left\{\frac{\partial v}{\partial x} + m\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) - m^2\frac{\partial u}{\partial y}\right\}\Bigg]. \end{aligned} \tag{10}$$

Kasner has shown that† as m takes on all values, df/dz_m traces out a circle, termed by Hedrick the Kasner circle for the point (x, y). From (10) we obtain‡

$$\left|\frac{df}{dz_m}\right| = \left\{\frac{1}{(1 + m^2)}(E + 2Fm + Gm^2)\right\}^{1/2}, \tag{11}$$

$$\operatorname{amp}\left(\frac{df}{dz_m}\right) = \tan^{-1}\left\{\frac{\dfrac{\partial v}{\partial x} + m\left(\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial x}\right) - m^2\dfrac{\partial u}{\partial y}}{\dfrac{\partial u}{\partial x} + m\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) + m^2\dfrac{\partial v}{\partial y}}\right\}. \tag{12}$$

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* For equation (9) and the latter system, see Hedrick, loc. cit., p. 78.

† Hedrick, loc. cit., pp. 80–81.

‡ See Hedrick, loc. cit., p. 78, for (11).

Now from (8), $E+2Fm+Gm^2=J(1+m^2)$. Hence we have the following theorem.

THEOREM 3. *In a contact direction* m,

$$\left|\frac{df}{dz_m}\right| = |J|^{1/2}. \tag{13}$$

If we apply the methods of the differential calculus to (11) and (12), we obtain the following result.

THEOREM 4. *The extremal values of* $|df/dz_m|$ *are* in the ortho (that is, the principal) directions, and the extremal values of* amp (df/dz_m) *are in the contact directions (when these are real).*

Denote the Kasner circle (for a given point (x, y)) by K. From the extremal properties of systems (8) and (9) we obtain the following theorem.

THEOREM 5. *If* O *is the origin and* Z *the center of the Kasner circle* K, *then the line through* O *and* Z *cuts* K *in points which give* df/dz_m *in the characteristic directions, and the points of tangency of the tangents to* K *from* O *(when* O *is not interior to* K*) give* df/dz_m *in the contact directions.*

From Theorems 3 and 5 we obtain the following corollary.

COROLLARY. *The common length of the tangents† from* O *to* K *(when they exist) is* $|J|^{1/2}$.

Let m_1, m_2 be a pair of dual directions. We may ask where the corresponding points df/dz_{m_1}, df/dz_{m_2} lie on K. The answer is given by the following theorem.

THEOREM 6. *Let* m_1, m_2 *be dual directions. Then the points* df/dz_m *for* $m=m_1$, m_2 *lie (on* K*) on a line through the origin. Conversely, points on* K *that are concurrent with* O *correspond to dual directions.*

To establish this, let us equate tan $\{\text{amp}(df/dz_m)\}$ to a constant k, using (12). This yields the equation

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* See Hedrick, loc. cit., p. 78.

† See Hedrick, loc. cit., p. 84.

$$m^2\left[k\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\right]+m\left[k\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\right]$$
$$+\left[k\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}\right]=0.$$

Denoting the two roots by m_1 and m_2, we have

$$m_1+m_2=\frac{-\left[k\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\right]}{k\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}},$$

$$m_1m_2=\frac{k\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}}{k\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}}.$$

If now we substitute these values in (3), we find, on using (6) and (7), that (3) is satisfied identically for all k. Hence the converse is true. But all points of K are thus paired off, and since a given direction has but *one* dual, the first half of the theorem is likewise true.

For all lines through O, meeting the Kasner circle K in points M_1, M_2, the product $OM_1\cdot OM_2=$constant. That constant must, by the Corollary* to Theorem 5, be $|J|$. Hence from Theorem 5 we deduce the next theorem.

THEOREM 6. *If m_1, m_2 are any two dual directions, then*

$$\left|\frac{df}{dz_{m_1}}\cdot\frac{df}{dz_{m_2}}\right|=|J|. \tag{14}$$

If we set $m=\tan\theta$, the Kasner circle can be written parametrically as follows:†

$$\frac{df}{dz_m}=\mathcal{D}[f]+\mathcal{P}[f]e^{-2i\theta}, \tag{15}$$

* Strictly speaking, the Corollary applies only when O is not interior to K; but the result can be shown to be true in all cases.

† Hedrick, loc. cit., p. 76.

where

$$(16)\qquad \begin{aligned} \mathcal{D}[f] &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+i\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right], \\ \mathcal{P}[f] &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+i\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]. \end{aligned}$$

Kasner has termed the quantity $\mathcal{D}[f]$, which is the center of the circle, the *mean derivative of f;* and indeed it is an average of df/dz_m for all points of K. There is a second point, $\mathcal{D}^*[f]$, which we believe may prove of use. It is obtained as follows. If we pair off points on K corresponding to dual directions, their (geometric) mean absolute value is $|J|^{1/2}$ (Theorem 6). Such a pair of points lie on a line through O, and if we average, arithmetically, the amplitudes (that is, inclinations) of such lines, we obtain the amplitude of the line through the center. We may therefore consider the quantity

$$(17)\qquad \mathcal{D}^*[f] = |J|^{1/2}e^{i\alpha},$$

where

$$\alpha = \text{amp}\,\mathcal{D}[f], \quad \tan\alpha = \frac{v_x - u_y}{u_x + v_y},$$

as being, in some sense, an average derivative. A simple reduction gives us

$$(18)\qquad \mathcal{D}^*[f] = \frac{2|J|^{1/2}}{(E+G+2J)^{1/2}}\mathcal{D}[f].$$

$\mathcal{D}^*[f]$ is thus defined for all points (x, y), where $E+G+2J\neq 0$. Now $E+G+2J=0$ when and only when $\mathcal{D}[f]=0$; which is to say, if and only if the point (x, y) is one at which the transformation (2) is *inversely-conformal*. Hence $\mathcal{D}^*[f]$ is defined whenever the K-circle does not have its center at the origin. And when the center is at the origin, $\mathcal{D}^*[f]$ is indeterminate.

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