PRIME ENDS AND INDECOMPOSABILITY*

BY N. E. RUTT†

1. *Introduction.* Under some circumstances the set of the prime ends‡ of a plane bounded simply connected domain includes one which contains all the boundary points of the domain. This paper will establish a sufficient condition for the existence of such a prime end, and also a necessary condition. One of the arguments to be given shortly will require a special sequence of simple closed curves. The following informal development will establish satisfactorily the existence and chief properties of the sequence.

Let γ be a simply connected bounded plane domain containing the continuum G and having boundary Γ. Let ϵ be any preassigned positive number. There exists a class G_1^ϵ of simple closed curves such that if C is any one of G_1^ϵ and c is any point of C, then§ $\gamma \supset i(C) \supset G$, and $d(c, \Gamma) < \epsilon$. There exists a subclass G_2^ϵ of G_1^ϵ such that if C is any one of G_2^ϵ and g_u is any point of $\Gamma + \gamma \cdot e(C)$ and g_v any point of γ such that $d(g_v, \Gamma) > \epsilon$, then $d(g_u, C) < \epsilon$ and $i(C) \supset g_v$. If $[p_h]$, $(h = 1, 2, \cdots, m)$, is any finite subset of the points of Γ, there is a subclass G_3^ϵ of G_2^ϵ such that, if C is any one of G_3^ϵ, then $e(C) \supset \sum p_h$. There is a subclass G_4^ϵ of G_3^ϵ such that if C is any one of G_4^ϵ, then $C \cdot \Gamma$ is a finite nonvacuous set of points $[c_g]$, $(g = 1, 2, \cdots, n)$, and if K is any component of $C - \sum c_g$, then $d(K) < \epsilon$. If $[P_{hk}]$, $(h = 1, 2, \cdots, m$ and $k = 1, 2)$, is a set of $2m$ arcs with end points $[p_{hk}, c_{hk}]$, respectively, such that $\gamma \supset \sum (P_{hk} - p_{hk})$, and if $p_{hk} = p_h$, $(k = 1, 2)$, for all values of h, and if $\gamma \cdot P_{hk} \cdot P_{h'k'} = 0$ for all admissible values of h, k, h', and k' except of course when $h = h'$ and $k = k'$;

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† Theorems 1 and 2 of this paper were obtained while the author was holding a National Research Fellowship.

‡ Defined by C. Carathéodory in his paper, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen, vol. 73 (1912), pp. 323–370.

§ If K is a point set then $c(K)$ represents its complement in space. If C is a simple closed curve in the plane then $i(C)$ represents its interior and $e(C)$ its exterior. Combining the symbols gives $ce(C) = C + i(C)$ and $ci(C) = C + e(C)$.

then there exists a subclass G_5^ϵ of G_4^ϵ such that, if C is any one of G_5^ϵ, then $i(C) \supset \sum c_{hk}$ and $C \cdot \sum P_{hk}$ is exactly $2m$ points. There is a subclass G_6^ϵ of G_5^ϵ such that, if C is any element of G_6^ϵ and K is any component of $C - \sum c_g$ containing a point of $\sum P_{hk}$, then K contains exactly two such points and is not adjacent on C to any component of $C - \sum c_g$ containing any points of $\sum P_{hk}$. It is to be observed that when $[p_{hk}]$ is vacuous, then $G_2 = G_3$ and $G_4 = G_5 = G_6$.

The existence of G_6^ϵ implies the existence of systems of simple closed curves now to be defined and hereafter to be referred to as of *type W* with respect to γ and G. When γ is any plane bounded connected and simply connected domain containing the continuum G and having boundary Γ, there exists an infinite sequence $[C_i]$, $(i = 1, 2, 3, \cdots)$, of simple closed curves related to a corresponding infinite sequence of positive numbers λ_i, $(\lambda_{i+1} < \lambda_i/2)$, in such a way that for each value of j: (1) the domain $i(C_j) \supset G$; (2) the domain $\gamma \supset i(C_j)$; (3) the set $\Gamma \cdot C_j$ is a nonvacuous finite point set $[c_{ji}]$, $(i = 1, 2, \cdots, n_j)$, and $\sum_{i=1}^{n_j} c_{ji} \cdot \sum_{m=1}^{j-1} \sum_{i=1}^{n_m} c_{mi} = 0$; (4) the components of $C_j - \sum_{i=1}^{n_j} c_{ji}$ are a set $[A_{ji}]$ of cuts of γ each of diameter less than λ_j; (5) the components of $C_m \cdot ci(C_j)$, $(m = 1, 2, \cdots, j-1)$, are a set of n_m mutually exclusive arcs $[C_{mi}^j]$ each of which contains precisely one point of $[c_{mi}]$, and $\sum_{i=1}^{n_h} C_{hi}^j \cdot \sum_{i=1}^{n_k} C_{ki}^j = 0$ if h and k are unequal values of m; (6) no two successive elements of the set of cuts $[A_{ji}]$ contain points of $\sum_{m=1}^{j-1} \sum_{i=1}^{n_m} C_{mi}^j$; (7) for any i, if $A_i \cdot \sum_{m=1}^{j-1} \sum_{i=1}^{n_m} C_{mi}^j \neq 0$, it is two points; and (8) if R is a point set such that $\gamma \cdot R \neq 0$, a number r exists such that, if $j > r$, then $i(C_j) \cdot R \neq 0$.

The existence of the sequence may be seen as follows. With $[p_h]$ vacuous, at least two points in $[c_g]$, and $\lambda = \lambda_1$ an arbitrary positive number, let C_1 be any element of G_6^ϵ. Assume that all curves of the set $[C_i]$ up to and including C_s have been constructed so as to possess as a set properties (1) to (7) inclusive. The next curve of the set will be constructed. Let $[p_h]$ be the set $\sum_{j=1}^{s} \sum_{i=1}^{n_i} c_{ji}$, $(h = 1, 2, \cdots, m$ and $m = \sum_{i=1}^{s} n_i)$. Let σ be the least positive number of the form $d(p_u, p_v)$, $(u$ and $v \leqq h)$. Let S_h, $(h = 1, 2, \cdots, m)$, be a circle with center p_h and radius less than $\sigma/3$ intersecting exactly one of the curves $C_1, C_2, \cdots, C_s$. For each value of h let P_h be the arc of $ce(S_h) \cdot \sum_{i=1}^{m} C_i$ which is maximal and contains p_h. Let η be $d(\Gamma, \sum_{i=1}^{s} C_i - \sum_{i=1}^{m} P_h)$.

Finally, with λ_{s+1} chosen less than $\lambda_s/2$ and less than η, and $[P_{hk}]$, $(h=1, 2, \cdots, m$ and $k=1, 2)$, chosen so that, for each value of h, the arc $P_h = P_{h1} + P_{h2}$, where P_{h1} and P_{h2} are arcs with only p_h in common, let C_{s+1} be any element of G_6^τ, $(\tau = \lambda_{s+1})$. The curves $C_1, C_2, \cdots, C_s, C_{s+1}$ have now been constructed in such a way as to possess all except the last of the properties listed. By continuation of the process an infinite sequence results which clearly has property (8).

2. *Sufficient Condition.* LEMMA 1. *Suppose* (1) *that* γ *is a plane bounded connected and simply connected domain containing a continuum* G *and having boundary* Γ, (2) *that* $[C_i]$ *is a collection of simple closed curves of type W with respect to* γ *and* G, *the general element* C_j *of which consists of the finite nonvacuous subset* $[c_{ji}]$ *of the points of* Γ *and the set* $[A_{ji}]$ *of cuts of* γ, (3) *that* $[A_i]$ *is any chain of cuts of* γ *each one of which for some value of j is included in the collection* $[A_{ji}]$, *and* (4) *that* $\mathfrak{P}$ *is a prime end of* γ *and is defined by the chain of cuts* $[B_i]$ *of* γ, *where there is a point of* Γ *every neighborhood of which contains all but a finite number of* $[B_i]$. *Then* (1) *the chain* $[A_i]$ *defines a prime end of* γ, *and* (2) *there is a chain of cuts of* γ *defining* $\mathfrak{P}$ *each one of which for some value of j belongs to* $[A_{ji}]$.

As $[A_i]$ is a chain, no more than one element of it can belong to the collection $[A_{ji}]$ for any given value of j. Thus $\lim_{i\to\infty} d(A_i)=0$, and so $[A_i]$ defines a prime end of γ.

On the other hand, owing to the conditions imposed upon $[B_i]$, there is a point b such that any domain containing b contains all but a finite number of $[B_i]$. There is a positive integer n_0 such that if $j>n_0$, then $C_j \not\supset b$. Accordingly there is a neighborhood η_0 of b containing no point of $ce(C_{n_0})$ and within it an element B_{n_0} of $[B_i]$. The cut B_{n_0} is separated in γ from $i(C_{n_0})$ by an element A_{n_0b} of $[A_{n_0i}]$. There is a positive number ρ_0 such that $\rho_0 < d(B_{n_0}, B_{n_0+1})$ and a subscript n_1, $\lambda_{n_1} < \rho_0$, such that $i(C_{n_1})$ contains points of B_{n_0+1}. As no element of $[A_{n_1i}]$ has points in both B_{n_0} and B_{n_0+1}, there are elements of $[A_{n_1i}]$ separated from $i(C_{n_0})$ in γ by B_{n_0}. There is a neighborhood η_1 of b such that $ce(C_{n_1})\cdot\eta_1=0$ and $\eta_1 \supset B_{n_1}$, where $n_1 > n_0+1$. But then $e(C_{n_1}) \supset B_{n_1}$, so there is an element A_{n_1b} of $[A_{n_1i}]$ which separates B_{n_1} and $i(C_{n_1})$ in γ, and this one separates B_{n_1} and B_{n_0} in γ, for it has no point in B_{n_0+1}. The process may be continued indefinitely, and the resulting chains of cuts B_{n_0}, $B_{n_1}, \cdots$ and A_{n_0b}, $A_{n_1b}, \cdots$ are equivalent. Thus there is a chain of cuts of γ defining $\mathfrak{P}$, any one of whose elements is included for some value of j in the collection $[A_{ji}]$.

THEOREM 1. *In order that the collection of the prime ends of the plane bounded connected and simply connected domain γ having boundary Γ should include one containing Γ, it is sufficient that Γ be indecomposable.*

Let G be any point of γ and $[C_i]$ be a collection of simple closed curves of type W with respect to γ and G. For every value of j the set $\gamma-\gamma\cdot\sum_{i=1}^{j}ce(C_j)$ is a finite set of domains $[\gamma_{ji}]$, some separated from G by one of $[A_{ji}]$ and the others by a cut which is the sum of a subarc of an element of $[A_{ji}]$ and a subarc of some other one of $[C_i]$. It is to be noted that each one of $[\gamma_{j+1,\,i}]$ is contained in one of $[\gamma_{ji}]$, that the boundary of the element γ_{ji} has as its subset in Γ a continuum Γ_{ji}, that $\Gamma=\sum\Gamma_{ji}$ for each value of j, and that if the subcollection of $[\gamma_{j+1,i}]$ contained in γ_{jk} is $[\gamma_{j+1,i}^{k}]$, then $\Gamma_{jk}=\sum\Gamma_{j+1,i}^{k}$.

Now as Γ is indecomposable, there is a specific element Γ_1^g of $[\Gamma_{1i}]$ which is identical with Γ. Moreover as $\Gamma_1^g=\sum\Gamma_{2i}^g$ there is a specific element Γ_2^g of $[\Gamma_{2i}^g]$ which is identical with Γ, and so on indefinitely. There arises in this way an infinite sequence of domains $\gamma_1, \gamma_2, \gamma_3, \cdots$, where for each value of j the domain γ_j is an element of $[\gamma_{ji}]$, the domain $\gamma_j\supset\gamma_{j+1}$, and $\gamma_j'=\Gamma_j+H_j$, where H_j is a cut of γ, if not contained in a single one of $[C_i]$, then the sum of an arc from one of $[C_i]$ and an arc from another.

If for two different subscripts, u and v, $(u<v)$, the sets $H_u\cdot C_j$ and $H_v\cdot C_j$ are nonvacuous (j being fixed), then, when $u\leqq m\leqq v$, the set $H_m\cdot C_j\neq 0$, since indeed $H_u\cdot C_j\supset H_m\cdot C_j\supset H_v\cdot C_j$. Accordingly if ($j$ being fixed), for indefinitely many values of u, the set $H_u\cdot C_j\neq 0$, then $H_u\cdot C_j\neq 0$ for all values of u subsequent to j. But in this case when $u>j$ the set $H_u\cdot e(C_j)$ belongs to an element B_u of $[A_{ui}]$. Moreover when $u>j$ the collection $[B_u]$ is a set of mutually exclusive arcs defining a prime end $\mathfrak{P}$ of γ. Since, for each of these values of u, the cut B_u separates γ_u from G, the prime end $\mathfrak{P}$ contains Γ.

On the other hand, if regardless of the value of j the set $H_u\cdot C_j\neq 0$ for at most a finite set of values of u, then beginning with any element H_{j_1} of $[H_j]$, there exists an element H_{j_2}

subsequent to H_{j_1} having no points in the members of $[C_i]$ in which H_{j_1} has points, an element H_{j_3} subsequent to H_{j_2} having no points in the members of $[C_i]$ in which $H_{j_1}+H_{j_2}$ has points, and so on indefinitely. Let B_n be an element of $[A_{j_n i}]$ in which H_{j_n} has points. Then $[B_n]$ is a collection of mutually exclusive cuts of γ forming a chain and defining a prime end $\mathfrak{P}$ of γ. Moreover as B_n separates G from γ_{j_n}, the prime end $\mathfrak{P}$ contains Γ.

3. *Necessary Condition.* LEMMA 2. *Suppose* (1) *that* γ *is a plane bounded connected and simply connected domain with boundary* Γ, (2) *that* $\mathfrak{P}$ *is a prime end of* γ *containing the subset* P *of* Γ *and defined by a chain of domains* $[\eta_i]$ *determined respectively by a chain of cuts* $[H_i]$ *of* γ *where there is only one point of* Γ *every neighborhood of which contains points of more than one of* $[H_i]$, *and* (3) *that* Q *is any proper subcontinuum of* P. *Then the set of the points of* $(\Gamma-Q)\cdot\sum\overline{H}_i$ *is infinite.*

Only the case in which $(\Gamma-Q)\cdot\sum\overline{H}_i$ is vacuous needs to be considered. Let δ be the complementary domain of $c(Q)$ containing all but the ends of each of $[H_i]$. Each of $[H_i]$ is then a cut of δ. Of the two domains into which H_1 divides δ, let δ_1 be the one containing all but the ends of H_2. Clearly then $\delta_1\supset\eta_1$. But H_2 is a cut of δ_1; of the two domains into which H_2 cuts δ_1, let δ_2 be the one which contains all except the ends of H_3. Then $\delta_2\supset\eta_2$. In general, for $n=3, 4, 5, \cdots$, clearly H_n is a cut of δ_{n-1}. Of the two domains into which H_n divides δ_{n-1}, let δ_n be the one which contains H_{n+1}. Thus $\delta_n\supset\eta_n$ for each value of n. It is readily seen that at most a finite number of $[\delta_i]$ are unbounded. Consider the sets $\prod\overline{\delta}_i$ and $\prod\overline{\eta}_i$. Since, for each value of i, $\delta_i\supset\eta_i$, then $\prod\overline{\delta}_i\supset\prod\overline{\eta}_i$. Now by supposition $\prod\overline{\eta}_i$ is the point set contained by $\mathfrak{P}$, that is, it is P, whereas $\prod\overline{\delta}_i\subset Q$. Thus the assumption that $(\Gamma-Q)\cdot\sum H_i$ is vacuous implies the contradiction $Q\supset P$.

LEMMA 3. *Suppose* (1) *that* γ *is a plane bounded connected and simply connected domain with boundary* Γ, *and* (2) *that* $\mathfrak{P}$ *is a prime end of* γ *containing the continuum* P *of* Γ *whose continuum of chief points is* P_c, *and* (3) *that* Q *is a proper subcontinuum of* P *contained in* P_c. *Then* Q *is a continuum of condensation of* Γ.

Let q be any point of Q. As q is a chief point of $\mathfrak{P}$ there is a chain of cuts $[H_i]$ of γ defining $\mathfrak{P}$, such that every neighborhood of q contains points of all but a finite number of $[H_i]$ and $\lim_{i\to\infty} d(H_i)=0$. Thus q is a limit point of every infinite subset of the end points of the elements of $[H_i]$. Since Q is a proper subset of P, the point set $(\Gamma-Q)\cdot\sum\overline{H}_i$ is infinite, and q is a limit point of this set. Thus $\overline{\Gamma-Q}\supset q$, as was to be proved.

THEOREM 2. *In order that the plane bounded connected and simply connected domain* γ *having boundary* Γ *should have a prime end* $\mathfrak{P}$ *containing* Γ *it is necessary that* Γ *should be either indecomposable or the sum of two indecomposable continua.*

Suppose that the prime end $\mathfrak{P}$ of γ contains Γ. Let $\mathfrak{P}$ be defined by the chain of cuts $[H_i]$ of γ where there is just one point of Γ every neighborhood of which contains points of more than one of $[H_i]$ and $\lim_{i\to\infty} d(H_i)=0$. Let P_c be the continuum of the chief points of $\mathfrak{P}$. Suppose first that $P_c=\Gamma$, and let Q be any proper subcontinuum of Γ. Then Q is a continuum of condensation of Γ by Lemma 3, and so Γ is indecomposable.

Suppose next that $P_c\neq\Gamma$, and that Γ is decomposable and the sum of D_0 and D_3, two proper subcontinua of it. The set P_c is thus a continuum of condensation of Γ. If $P_c\supset D_0$ then $D_3\supset\Gamma-P_c$, and $D_3\supset\overline{\Gamma-P_c}=\Gamma$, which would mean that $D_3\equiv\Gamma$; thus $P_c\not\supset D_0$ and $P_c\not\supset D_3$. In addition $P_c\cdot D_0\neq 0$ and $P_c\cdot D_3\neq 0$, for if it is supposed that one of these is vacuous then Lemma 2 is contradicted, as all but a finite number of $[H_i]$ will necessarily have both of their end points in D_0 or D_3 in consequence. Accordingly $\Gamma-P_c\not\subset D_0$, and so $\Gamma-(P_c+D_0)=Q_3\neq 0$, where $D_3\supset Q_3$, and similarly $\Gamma-(P_c+D_3)=Q_0\neq 0$, where $D_0\supset Q_0$.

Suppose now that D_0 is expressible as the sum of its two proper subcontinua D_1 and D_2. Consider the possibility that neither D_1 nor D_2 contains Q_0. In this case $\Gamma-(P_c+D_1+D_2)$ is the nonvacuous set Q_3, also $\Gamma-(P_c+D_1+D_3)$ is a nonvacuous subset Q_2 of D_2, and $\Gamma-(P_c+D_2+D_3)$ is a nonvacuous subset Q_1 of D_1. As $D_0\cdot P_c\neq 0$, either D_1 or D_2 contains a point of P_c. If $D_1\cdot P_c=0$, then D_2+D_3 is a continuum containing P_c and a contradiction of Lemma 2 speedily results. Thus $D_1\cdot P_c\neq 0$ and $D_2\cdot P_c\neq 0$. But then, either when one of the three continua $D_1+P_c+D_2$, $D_1+P_c+D_3$, and $D_2+P_c+D_3$, or when none of the three, contains both ends of infinitely many of $[H_i]$, there follows a contradiction of Lemma 2, so it is not possible for neither D_1 nor D_2 to contain Q_0.

Suppose D_0 replaced by D_1, where D_1 is a subcontinuum of D_0 irreducible about Q_0. Then $\Gamma = D_1 + D_3$, and the properties of D_0 and D_3 already derived are possessed also by D_1 and D_3. But if D_1 is decomposable, then neither of any two of its proper subcontinua of which it is the sum can contain Q_0, as D_1 is irreducible about Q_0, whereas, if neither of them contains Q_0, then the contradiction above appears.

So D_1 is indecomposable, and if D_2 is a subcontinuum of D_3 irreducible about $\Gamma - (P_c + D_1)$, then D_2 is also indecomposable. As $\Gamma = D_1 + P_c + D_2$, then $\Gamma - P_c = (D_1 + D_2) - (D_1 + D_2) \cdot P_c$; and as $\overline{\Gamma - P_c} = \Gamma$, then $\overline{(D_1 + D_2) - (D_1 + D_2) \cdot P_c} = \Gamma$. This means that $D_1 + D_2 \supset P_c$, so $\Gamma = D_1 + D_2$, and the theorem is proved.

4. *Remarks and Generalizations.* The sufficient condition of §2 is not necessary, as the following example shows. Let the curvilinear quadrilateral K_1 with vertices P, Q, R, and S consist of the points (x, y) where $x \geqq 0$ and $1 \leqq x^2 + y^2 \leqq 4$; and the curvilinear quadrilateral K_2, also with vertices P, Q, R, and S, consist of the points (x, y) where $x \leqq 0$ and $1 \leqq x^2 + y^2 \leqq 4$. In each case let the four vertices be $P(0, 2)$, $Q(0, 1)$, $R(0, -1)$, and $S(0, -2)$. Upon the square $A(0, 0)$, $B(0, 1)$, $C(1, 1)$, and $D(1,0)$ suppose the indecomposable continuum of Brouwer* inscribed. Let K_2 be homeomorphic with $ABCD$ in such a way that D coincides with Q, C with P, B with S, and A with R. Also let K_1 be symmetric to K_2 with respect to the y axis. A domain γ results containing the origin of coordinates and bounded by a bounded decomposable continuum Γ which is the sum of two of Brouwer's indecomposable continua. The set contains PQ, and PQ belongs to a prime end of γ which contains Γ.

The necessary condition of §3 is not sufficient, as the following example shows. Let K_2 be homeomorphic with $ABCD$ as above. Let K_1 be homeomorphic with $ABCD$ also, in such a way that A coincides with Q, B with P, C with S, and D with R. A bounded domain results having a boundary which is the

* As in the figure on p. 228 of vol. 8 of the Fundamenta Mathematicae. See also Brouwer, Mathematische Annalen, vol. 68, p. 423. Brouwer's diagram includes more than is used in the examples of this paper.

sum of two of Brouwer's indecomposable continua, and no prime end of this domain contains both of these continua.

It may be remarked that Lemma 2 is not true if $\lim_{i\to\infty} d(H_i)\neq 0$, and neither Lemma 2 nor Lemma 3 is true if Q is not a proper subcontinuum of P. Lemma 3 implies the following theorem, of interest in itself, which was proved in the derivation of Theorem 2.

THEOREM 3. *If the plane bounded connected and simply connected domain* γ *has a prime end whose set of chief points coincides with* Γ, *the boundary of* γ, *then* Γ *is indecomposable.*

The theorem of §2 may be generalized.

LEMMA 4. *Suppose* (1) *that the point set* D *is a plane bounded indecomposable continuum,* (2) *that the point set* K *is a plane bounded continuum such that* $D\cdot c(K)\neq 0$, *and that* $\overline{D\cdot c(K)}\neq D$, *and* (3) *that* γ *is a component of* $c(D+K)$. *Then* $\overline{\gamma}\not\supset D$.

Suppose indeed that γ is a component of $c(D+K)$ such that $\overline{\gamma}\supset D$. Let δ be the component of $c(K)$ which contains γ. Then $\delta\cdot D=c(K)\cdot D$ and is a non-vacuous set. Since $\overline{D\cdot c(K)}\neq D$, the set $\delta\cdot D$ can not contain entirely any of the composants of D. Consequently $\delta\cdot D$ is a collection of components no one of which contains points of more than one composant of D. Thus the components of $D\cdot c(K)$ are an uncountable collection $[D_\alpha]$, and as $\overline{\gamma}\supset\sum D_\alpha$, this collection is orderable.*

It is moreover readily seen that no two of $[D_\alpha]$ belonging to one composant of D can separate two which belong to some other composant, so that the subcollection of $[D_\alpha]$ included in any composant of D is an interval of the collection $[D_\alpha]$. Let D_k be any element of $[D_\alpha]$ and contain the point k. The point k is a limit of every composant of D, and so is a limit of each of an uncountable set of series of $[D_\alpha]$ each contained in a different composant of D. This is a contradiction, as the set of series in question is uncountable in number and as no two of them are concurrent.†

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* N. E. Rutt, *On certain types of plane continua*, Transactions of this Society, vol. 33 (1931), p. 806.

† N. E. Rutt, *Concurrence and uncountability*, this Bulletin, vol. 39 (1933), Corollary 1, p. 299.

THEOREM 4. *If the boundary Γ of the plane bounded connected and simply connected domain γ contains an indecomposable continuum D, there is a prime end of γ which contains D.*

Here, as in the development of Theorem 2, for each value of j the set $\Gamma=\sum\Gamma_{ji}$. Consequently $\sum\Gamma_{ji}\supset D$. If for each of these $\overline{c(\Gamma_{ji})\cdot D}\supset D$, then the set $\sum\Gamma_{ji}\cdot D$ is nowhere dense in D and $[\Gamma_{ji}]$ does not cover D. But as none of $[\Gamma_{ji}]$ can have $\overline{c(\Gamma_{ji})\cdot D}$ $\not\supset D$ unless $D\cdot c(\Gamma_{ji})=0$, in view of Lemma 4, there must for every value of j be one of $[\Gamma_{ji}]$ which contains D. The proof now follows lines almost identical with those of Theorem 2.

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PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVES

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In a fundamental paper* on the projective differential geometry of curves, L. Berzolari obtained canonical expansions representing a curve C immersed in a linear space S_n in a neighborhood of one of its points P_0. The vertices of the coordinate simplex yielding Berzolari's canonical form are covariantly related to the curve, while the unit point may be any point of the rational normal curve Γ which osculates C at P_0. It is the purpose of the present paper to define a covariant point on Γ which can be chosen as a unit point so as to produce final canonicalization of the power series expansions of Berzolari.

It will be observed that the usual methods of defining a point on Γ for the cases $n=2$ and $n=3$ depend on configurations† that do not possess suitable analogs in n-space. Hence it appeared for some time that the problem called for different procedures in spaces of different dimensionality. Special devices

* L. Berzolari, *Sugli invarianti differenziali proiettivi delle curve di un iperspazio*, Annali di Matematica, (2), vol. 26 (1897), pp. 1–58.

† E. P. Lane, *Projective Differential Geometry of Curves and Surfaces*, pp. 12–27.