ON A CERTAIN NON-LINEAR ONE-PARAMETER SYSTEM OF HYPERSURFACES OF ORDER n IN r-SPACE

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Consider a linear ∞^{ρ}-system, where

$$\rho \leqq \frac{(n+r)!}{n!\,r!} - 1,$$

of hypersurfaces of order n, which may have σ base points, in an r-space, S_r. Denote this system by $|W|$.

Now let $\nu_1+\nu_2+\cdots+\nu_t$ projectively related curves

$$C_{11}, C_{12}, \cdots, C_{1\nu_1}, C_{21}, C_{22}, \cdots, C_{2\nu_2}, \cdots, C_{t\nu_t}$$

of orders

$$n_{11}, n_{12}, \cdots, n_{1\nu_1}, n_{21}, n_{22}, \cdots, n_{2\nu_2}, \cdots, n_{t\nu_t},$$

respectively, and all of genus p, be given in the same r-space S_r. To a point on any one of the curves corresponds a definite point on each of the other curves. We assume that none of the given curves passes through any of the σ base points of $|W|$ and that none of the intersections, if there be any, of any two of the curves is a self-corresponding point. Let $P_{11}, P_{12}, \cdots, P_{t\nu_t}$ be a set of corresponding points, the point P_{ij_i} being on the curve C_{ij_i}, $(i=1, 2, \cdots, t; j_i=1, 2, \cdots, \nu_i)$. If

$$\text{(I)} \qquad \nu_1 + 2\nu_2 + \cdots + t\nu_t = \rho \leqq \frac{(n+r)!}{n!r!} - 1,$$

there is one and only one hypersurface of the system $|W|$ such that $1, 2, \cdots, t$ of the points of its intersection with each of the ν_1 curves C_{1j_1}, ν_2 curves $C_{2j_2}, \cdots, \nu_t$ curves C_{tj_t}, will coincide with $P_{1j_1}, P_{2j_2}, \cdots, P_{tj_t}$, respectively. Denote such a hypersurface by V_{r-1}^n. As the corresponding points describe their respective curves, V_{r-1}^n describes a non-linear one-parameter system, $\{V\}$, of hypersurfaces of order n in S_r. In this paper we propose to determine the number, N_0, the order of the system, of the hypersurfaces of the system passing through a given point and also the number, N_k, of those tangent to a given k-space for $k=1, 2, \cdots, r$. The symbol N_r means the number of the hypersurfaces that have each a node.

In the following, we shall give two determinations of the number N_0: the one by the use of the theory of correspondence and the other by the aid of the following known proposition.*

(A) Let there be given q varieties $V_{x_1}^{m_1}, V_{x_2}^{m_2}, \cdots, V_{x_q}^{m_q}$ of orders $m_1, m_2, \cdots, m_q$, respectively, such that $V_{x_i}^{m_i}$ is the locus of ∞^1 (x_i-1)-spaces. If there exists a one-to-one correspondence between the elements of these varieties, then the locus of the ∞^1 $(x_1+x_2+\cdots+x_q-1)$-spaces determined by corresponding elements is a $V_{x_1+x_2+\cdots+x_q}$ of order $m_1+m_2+\cdots+m_q$.

We now determine N_0 by the theory of correspondence. We commence with the case $\nu_1=\rho=2$, $\nu_2=\nu_3=\cdots=\nu_t=0$. The system $\{V\}$ now consists of those hypersurfaces of the net $|W|$ which pass through pairs of corresponding points on the two given curves C_{11}, C_{12}. The desired number is the number of hypersurfaces of $\{V\}$ passing through a given point, say A. Let us make a hypersurface W_{r-1}^n of $|W|$ pass through A and a point P_{11} of C_{11}. This W_{r-1}^n meets C_{12} in nn_{12} points $Q_{12}, Q_{12}', \cdots$. If one of these points happens to coincide with the point P_{12} corresponding to P_{11} on C_{11}, then W_{r-1}^n is a V_{r-1}^n of $\{V\}$. In general, this does not happen. Now pass another hypersurface $W_{r-1}'^n$ of $|W|$ through A and one of the points $Q_{12}, Q_{12}', \cdots$, say Q_{12}. This $W_{r-1}'^n$ meets C_{11} in nn_{11} points $P_{11}, P_{11}', \cdots$, to which correspond nn_{11} points $P_{12}, P_{12}', \cdots$, on C_{12}. We see that we have thus established a correspondence on the curve C_{12} such that to each of the points $Q_{12}, Q_{12}', \cdots$ correspond nn_{11} points P_{12}, $P_{12}', \cdots$ and to each of the latter correspond nn_{12} points of the former. If a united point occurs, then the two hypersurfaces $W_{r-1}^n, W_{r-1}'^n$ become coincident with a V_{r-1}^n of $|V|$. The correspondence being obviously of valence zero, the number of united points, and therefore the order of $\{V\}$, is $n(n_{11}+n_{12})$.

Suppose now $\nu_1=\rho=3$, $\nu_2=\nu_3=\cdots=\nu_t=0$. Choose a W_{r-1}^n of the web $|W|$ that passes through a given point A and a pair

* A full discussion of this proposition is found in Edge, *On the quartic developable*, Proceedings of the London Mathematical Society, (2), vol. 33, pp. 52–65. The statement above is quoted verbatim from B. C. Wong, *On the number of stationary tangent S_{t-1}'s to a V_k^n in an S_{tk+k-1}*, this Bulletin, vol. 39 (1933), pp. 608–610.

of corresponding points P_{11}, P_{12} on C_{11}, C_{12}. This W_{r-1}^{n} meets the third given curve C_{13} in nn_{13} points Q_{13}, Q_{13}', $\cdots$, none of which, in general, coincides with the point P_{13} corresponding to P_{11} and P_{12}. Through each of these points Q_{13}, Q_{13}', $\cdots$ there are, according to the result just found, $n(n_{11}+n_{12})$ hypersurfaces each containing A and a pair of points P_{11}, P_{12}. Then on C_{13} there are $n(n_{11}+n_{12})$ points P_{13}, P_{13}', $\cdots$ corresponding to as many pairs on C_{11}, C_{12}. Now we have on C_{13} an $(n_{11}n+n_{12}n, n_{13}n)$-correspondence of valence zero between the points P_{13}, P_{13}', $\cdots$ and the points Q_{13}, Q_{13}', $\cdots$. The number of united points, and therefore the number of the hypersurfaces of $|W|$ passing through A and a set of corresponding points on C_{11}, C_{12}, C_{13}, is $n(n_{11}+n_{12}+n_{13})$.

If we continue reasoning in this manner, we shall find that, for the case $\nu_1=\rho$, $\nu_2=\nu_3=\cdots=\nu_t=0$, the order of $\{V\}$ is $n(n_{11}+n_{12}+\cdots+n_{1\nu_1})$ or $n\sum_{j_1=1}^{\nu_1}n_{1j_1}$.

Now suppose $\nu_1=\rho-2$, $\nu_2=1$. Then the system $\{V\}$ consists of all those hypersurfaces of the ∞^{ρ}-system $|W|$ which contain a set of corresponding points P_{1j_1}, $(j_1=1, 2, \cdots, \nu_1)$, on the ν_1 curves C_{1j_1} and have a contact with C_{21} at the point P_{21} corresponding to P_{1j_1}. Select a W_{r-1}^{n} of $|W|$ passing through a given point A and a fixed set of points P_{1j_1} and having a point of contact with C_{21}. Since a pencil of hypersurfaces of order n contains $2(n_{21}n-1+p)$ members tangent to a given curve of order n_{21} and genus p, there are $2(n_{21}n-1+p)$ such hypersurfaces and hence there are as many points of contact T_{21}, T_{21}', $\cdots$ on C_{21}. None of these, in general, coincides with P_{21}. Now pass a hypersurface $W_{r-1}'^{n}$ of $|W|$ through the points P_{1j_1} on the curves C_{1j_1} tangent to C_{21} at one of the points T_{21}, T_{21}', $\cdots$, say T_{21}. There are, according to the result of the preceding paragraph, $n\sum_{j_1=1}^{\nu_1}n_{1j_1}$ such hypersurfaces giving rise to as many sets of corresponding points P_{1j_1}, P_{1j_1}', $\cdots$, to which correspond as many points P_{21}, P_{21}', $\cdots$, on C_{21}. Thus, we have established an $[n\sum_{j_1=1}^{\nu_1}n_{1j_1}, 2(n_{21}n-1+p)]$-correspondence also of valence zero between the points P_{21}, P_{21}', $\cdots$, and the points T_{21}, T_{21}', $\cdots$, on C_{21}. The number of united points in this correspondence which gives the order of $\{V\}$ is therefore $n\sum_{j_1=1}^{\nu_1}n_{1j_1}+2(n_{21}n-1+p)$.

These particular cases are sufficient to indicate the method used. Reasoning in exactly the same manner for all the different values of the ν's satisfying (I), we find the general result

$$(1)\qquad N_0 = \sum_{j_1=1}^{\nu_1} n_{1j_1}n + 2\sum_{j_2=1}^{\nu_2}(n_{2j_2}n - 1 + p)$$

$$+ 3\sum_{j_3=1}^{\nu_3}(n_{3j_3}n - 2 + 2p) + \cdots + t\sum_{j_t=1}^{\nu_t}[n_{tj_t}n + (t-1)(p-1)],$$

or

$$N_0 = \sum_{i=1}^{t} i\sum_{j_i=1}^{\nu_i}[n_{ij_i}n + (i-1)(p-1)],$$

where $i[n_{ij_i}n+(i-1)(p+1)]$ is the number of hypersurfaces of order n of an ∞^{i-1}-system of hypersurfaces such that i of the points of intersection of each of them with a given curve of order n_{ij_i} are coincident.

Now we determine N_0 by the aid of (A). Let the hypersurfaces of $|W|$, which may have σ base points, represent upon S_r an r-dimensional variety $\Phi_r^{n^r-\sigma}$ of order $n^r-\sigma$ in a ρ-space S_ρ. The $\nu_1+\nu_2+\cdots+\nu_t$ given curves, none of which is supposed to pass through any of the σ base points, are the images of curves Γ_{ij_i} of order $n_{ij_i}n$ on $\Phi_r^{n^r-\sigma}$ whose points are also in a one-to-one correspondence. Let R_{11}, R_{12}, $\cdots$ be a set of corresponding points, the point R_{ij_i} being on the curve Γ_{ij_i}. Corresponding to a hypersurface V_{r-1}^n of the system $\{V\}$ is a section $\Theta_{r-1}^{n^r-\sigma}$ of $\Phi_r^{nr-\sigma}$ by a $(\rho-1)$-space which contains a set of points R_{1j_1} on the curves Γ_{1j_1}, a set of tangent lines at the points R_{2j_2} on the curves Γ_{2j_2}, a set of osculating planes at the points R_{3j_3} on the curves Γ_{3j_3}, $\cdots$. The ∞^1 $(\rho-1)$-spaces of the nature just described form an ∞^1-system to which corresponds our system $\{V\}$ of hypersurfaces. By applying (A) we find that the order of the system of $(\rho-1)$-spaces is, since the i-dimensional developable of the curve Γ_{ij_i} is of order $i[n_{ij_i}n+(i-1)(p-1)]$, the same as (1). Now through a given point A' which may be, without loss of generality, placed upon $\Phi_r^{n^r-\sigma}$, pass the same number of $(\rho-1)$-spaces of the system and each such $(\rho-1)$-space intersects $\Phi_r^{n^r-\sigma}$ in a $\Theta_{r-1}^{n^r-\sigma}$ passing through A' to which corresponds a V_{r-1}^n of $\{V\}$ passing through a given point A, the image of A'. Thus, the determination is complete.

Hitherto we have assumed that none of the given curves passes through any of the base points of $|W|$ and that none of the intersections, if there be any, of any two of the curves is a self-corresponding point. If, however, a curve C_{ij_i} passes through one of the base points, we must deduct i, and if any two whatever of the curves intersect in a self-corresponding point, we must deduct unity from the general value of N_0 which we have just derived.

As an example consider a linear ∞^6-system $|K|$ of quartic curves in a plane ϕ with 8 base points. Let three projectively related cubic curves γ^3, γ'^3, γ''^3 of genus unity be given in the plane, none of the intersections of the curves being a self-corresponding point. Select a quartic of $|K|$ such that one of its intersections with γ^3 coincides at P, two of its intersections with γ'^3 coincide at P', and three of its intersections with γ''^3 coincide at P'', where P, P', P'' are a set of corresponding points. There are ∞^1 such quartic curves forming a non-linear pencil, $\{C\}$. Now the quartics of $|K|$ represent upon ϕ a surface Φ^8 of order 8 in S_6 upon which lie three projectively related curves Γ^{12}, Γ'^{12}, Γ''^{12} all of order 12 and genus 1, of which γ^3, γ'^3, γ''^3 are the images in ϕ. The locus of tangent lines of Γ'^{12} is of order 24 and the locus of osculating planes of Γ''^{12} is of order 36. Let R, R', R'' be a set of corresponding points on the curves. Then we say that the tangent t' to Γ'^{12} at R' and the osculating plane π'' to Γ''^{12} at R'' correspond to the point R on Γ^{12}. The 5-spaces determined by R, t', π'' will describe an ∞^1-system of 5-spaces such that $N_0=72$ of them pass through a given point A' which may be placed on Φ^8. Therefore the system $\{C\}$ of quartic curves contains 72 members passing through a given point A of ϕ.

Suppose the curve γ^3 passes through a base point of $|K|$. Then the corresponding curve Γ^{12} on Φ^8 is composed of a Γ^{11} and a line. Discarding the line or deducting unity, we have $N_0=71$. If γ'^3 alone contains a base point, the corresponding curve Γ'^{12} degenerates into a line, to be disregarded, and a Γ'^{11} whose developable surface is of order 22. Therefore, we deduct 2 and now $N_0=70$. Finally, let γ''^3 alone go through a base point. The curve Γ''^{12} on Φ^8 is made up of a line, also to be disregarded, and a Γ''^{11} the locus of whose osculating planes is of order 33. Deducting 3, we now have $N_0=69$.

Now let one of the intersections of γ^3, γ'^3 be a self-corresponding point. Then Γ^{12}, Γ'^{12} also have a self-corresponding point $R\equiv R'$ to which corresponds the point R'' on Γ''^{12}. There is a linear pencil of 5-spaces passing through the tangent line t' at the self-corresponding point $R\equiv R'$ and containing the osculating plane of Γ''^{12} at R''. Disregarding this pencil, we deduct 1 and the result is $N_0=71$. If γ'^3, γ''^3 have a self-corresponding point in common, then one of the intersections of Γ'^{12}, Γ''^{12} is a self-corresponding point $R'\equiv R''$ to which corresponds a point R of Γ^{12}. There is a linear pencil of 5-spaces determined by R and the tangent line to Γ'^{12} at R' and the osculating plane of Γ''^{12} at $R''\equiv R'$. Deducting 1, we have $N_0=71$.

We shall next proceed to determine N_k, the number of the hypersurfaces of $\{V\}$ tangent to a given k-space in S_r. We find it convenient to use the following method. We set up a one-to-one correspondence between the points of a ρ-space S_ρ and the hypersurfaces of the ∞^ρ-system $|W|$. Corresponding to the $\infty^{\rho-1}$ hypersurfaces of $|W|$ that pass through a given point A are the $\infty^{\rho-1}$ points of a $(\rho-1)$-space $S_{\rho-1}$ of S_ρ, and corresponding to the hypersurfaces of $\{V\}$ are the points of a curve Δ. Since there are given by (1), as we have seen, N_0 hypersurfaces of $|W|$ passing through A and belonging to $\{V\}$, there must be N_0 points of S_ρ common to $S_{\rho-1}$ and Δ. Hence Δ is of order N_0.

Let a k-space S_k be given in S_r. Contact being one condition, there are $\infty^{\rho-1}$ hypersurfaces of $|W|$ tangent to S_k, and to these contact hypersurfaces correspond $\infty^{\rho-1}$ points of a locus $\Sigma_{\rho-1}^M$ in S_ρ. By the methods of analytic geometry we find without difficulty that the order M of $\Sigma_{\rho-1}^M$ is $M=(k+1)(n-1)^k$. All those hypersurfaces of $|W|$ belonging to $\{V\}$ and tangent to S_k are given by all those points of S_ρ common to Δ^{N_0} and $\Sigma_{\rho-1}^M$. Therefore, the number of hypersurfaces of $\{V\}$ tangent to S_k is the number of the points in which Δ^{N_0} intersects $\Sigma_{\rho-1}^M$ and is therefore equal to $N_k=MN_0=(k+1)(n-1)^kN_0$.

For $k=1, 2$, then, $N_1=2(n-1)N_0$ and $N_2=3(n-1)^2N_0$ are, respectively, the number of hypersurfaces of $\{V\}$ tangent to a given line and the number of those tangent to a given plane. If $k=r$, we have $N_r=(r+1)(n-1)^rN_0$ members of the system that have each a node.

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