TRANSFINITE SUBGROUP SERIES*

BY GARRETT BIRKHOFF

1. *Summary*. This note contains a proof that the Theorem of Jordan-Hölder can be extended to the case of any series of normal subgroups or, more generally, to the case of what we shall call "T-invariant" subgroups well-ordered in the direction of increasing subgroups. An example is given showing that the replacement of "increasing" by "decreasing" in the preceding sentence renders the proposition false.

Finally, the situation as regards the subgroup-series of compact topological groups homeomorphic with subsets of Cartesian n-space is clarified by two superficial observations.

2. *Definitions and Notation*. Let G be any group; H and K any two subgroups of G. We shall write $H \cap K$ for the *meet* or cross-cut of H and K, and $H \cup K$ for the subgroup generated by the *join* of H and K. The statements $H<K$ and $K>H$ mean that H is contained in, but is different from, K; $H \nless K$ and $K \ngtr H$ mean that $H<K$ is false. The statement $H \supset K$ means H includes K.

Now let A be the group of all automorphisms, and A_I the subgroup of the inner automorphisms of G, and let T be any subgroup of A containing A_I. The subgroup H will be called *T-invariant* if and only if it is carried into itself under every automorphism of T. It is certain that any T-invariant subgroup is normal.

By a *T-series* of G we shall mean† any set Σ of T-invariant subgroups T_i of G with the two properties:

(i) If $i \neq j$, then either $T_i < T_j$ or $T_i > T_j$.

(ii) To every T-invariant subgroup X of G corresponds a $T_i \in \Sigma$ such that $T_i \nless X$ and $T_i \ngtr X$.

By a *well-ordered ascending* (*well-ordered descending*) T-series of G is meant one in which every subset has a least (greatest) term.

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† The cases $T = A_I$ and $T = A$ yield under these definitions normal subgroups and chief series, and characteristic subgroups and characteristic series. The cases $A_I < T < A$ yield generalizations.

3. *Extension of the Jordan-Hölder Theorem.* We shall want to use the following rather simple lemma.

LEMMA. *If G is any group, while S, T, and U are T-invariant subgroups of G such that S is a largest T-invariant subgroup of G in T, then either $S \cup U = T \cup U$ or $S \cup U$ is a largest T-invariant subgroup of G in $T \cup U$.*

Suppose the contrary, that G contained a T-invariant subgroup W satisfying $S \cup U < W < T \cup U$. Consider $W \cap T$; evidently $S \subset W \cap T \subset T$, whence $W \cap T = S$ or $W \cap T = T$. But if $W \cap T = T$, then $W \supset T$ as well as $W \supset U$, so that $W \supset T \cup U$ contrary to hypothesis. While if $W \cap T = S$, then since $U \subset W$, $S \cup U = (W \cap T) \cup U = W \cap (U \cup T) = W$, again contrary to hypothesis.*

We are now in a position to prove the following theorem.

THEOREM 1. *Let Σ: $1 = T_1 < T_2 < T_3 < \cdots < T_m = G$ and Σ': $1 = T_1' < T_2' < T_3' < \cdots < T_n' = G$ be any two well-ordered ascending T-series† of G. Then we can establish a $(1,1)$ correspondence between the T_{i+1}/T_i and the T_{j+1}'/T_j', such that corresponding factor-groups are isomorphic under an isomorphism preserved under every automorphism performed by T.*

For let $T_{j(i)}'$ be the first term of Σ' satisfying $T_{j(i)}' \cup T_i = T_{j(i)}' \cup T_{i+1}$. Then $j(i)$ is evidently single-valued and defined for all i.

Now $j(i)$ is not a limit-number. For since to contain one element of $T_{i+1} - T_i$ and all of T_i would be‡ to contain all of T_{i+1}, we can be sure that§

$$T_j' \cup T_i = \lim_{k \to j} T_k' \cup T_i = \sum_{k<j} T_k' \cup T_i$$

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* We are using the fundamental combinatorial formula for normal subgroups, that if $A \subset C$, then $A \cup (B \cap C) = (A \cup B) \cap C$. See Theorem 26.1 of my paper *On the combination of subalgebras*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 441–464.

† Here m and n are of course finite or transfinite ordinals.

‡ Since every $(T_k \cup T_i) \cap T_{i+1} = T(i, k)$ is a T-invariant subgroup of G, we must exclude the hypothesis $T_i < T(i, k) < T_{i+1}$.

§ If $S_1 \subset S_2 \subset S_3 \subset \cdots$, then $S_1 \cup S_2 \cup S_3 \cup \cdots = S_1 + S_2 + S_3 + \cdots$ for subalgebras of any algebra whose operations act only on finite sets of elements; for example, groups, rings, and lattices.

contains no elements of $T_{i+1}-T_i$. That is, $j(i)-1$ surely exists.

But $T_{j(i)-1} \cup T_i < T'_{j(i)-1} \cup T_{i+1}$ is by the Lemma a largest T-invariant subgroup of G in $T'_{j(i)} \cup T_i = T'_{j(i)} \cup T_{i+1}$. And we know that $T'_{j(i)-1} \cup T_{i+1} \subset T'_{j(i)} \cup T_{i+1}$; therefore $T'_{j(i)-1} \cup T_{i+1} = T'_{j(i)} \cup T_{i+1}$. That is, T_{i+1} is reciprocally the first term T_k of Σ such that $T_k \cup T'_{j(i)-1} = T_k \cup T'_{j(i)}$.

This establishes a (1, 1) correspondence between the T_{i+1}/T_i and the T'_{j+1}/T'_j. Since under this correspondence the association of each coset of T_{i+1}/T_i or T'_{j+1}/T'_j with that coset of $T_{i+1} \cup T'_{j+1}/T_i \cup T'_j$ containing it defines an isomorphism preserved under every automorphism of T, we have proved Theorem 1.

4. *Simple Counter-Examples*. Let G be the enumerable cyclic group generated by a single element g. Let A_i and B_i denote the (normal) subgroups of G generated by g^{2^i} and g^{3^i}, respectively. It is entirely evident that

$$G > A_1 > A_2 > A_3 > \cdots ; 1 \qquad \text{and}$$

$$G > B_1 > B_2 > B_3 > \cdots ; 1$$

are chief (and composition series in the natural sense of the word.* Yet the first contains only factor-groups of order two, and the second only those of order three. There results the following theorem.

THEOREM 2. *The enumerable cyclic Abelian group has well-ordered descending chief series which do not satisfy the theorem of Jordan-Hölder.*

We must not assume that because one T-series of a group is well-ordered in the direction of increasing subgroups, all of its T-series are. Take the enumerable Abelian group G generated by elements $a_1, a_2, a_3, \cdots$ of order two. Let S_i denote the (normal) subgroup generated by $a_1, \cdots, a_i$, and T_i the (normal) subgroup generated by $a_{i+1}, a_{i+2}, a_{i+3}, \cdots$. Then the S_i and the T_i (with I and G thrown in) constitute a counter-example.

* See, for instance, O. Schreier, *Über den Jordan-Hölderschen Satz*, Hamburg Abhandlungen, vol. 6, pp. 300–302. Since the present paper was written, Schreier's proof has been improved by H. Zassenhaus, *Zum Satz von Jordan-Holder-Schreier*, Hamburger Abhandlungen, vol. 10 (1934), pp. 106–109.

5. *Compact Topological Groups.* The following theorem is loosely related to Theorem 1.

THEOREM 3. *Let G be any compact topological group whose manifold is homeomorphic with a subset of Cartesian n-space. Then any series of closed subgroups of G can be well-ordered in the direction of increasing subgroups.*

For the different group nuclei* are at most $(n+1)$ in number. And the index of the subgroup generated by any one of these nuclei in any larger closed subgroup having the same nucleus is finite.

But if we restrict ourselves to *closed* T-invariant subgroups, then the proof of Theorem 1 breaks down. For consider the additive group of residues modulo unity. The subgroups generated by 1/2, 1/4, 1/8, $\cdots$ form one chief series, and those generated by 1/3, 1/9, 1/27, $\cdots$ a second one, and yet the two have not a single factor-group in common.

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LOCI OF m-SPACES JOINING CORRESPONDING POINTS OF $m+1$ PROJECTIVELY RELATED n-SPACES IN r-SPACE†

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Let $m+1$ n-spaces $S_n^{(1)}, S_n^{(2)}, \cdots, S_n^{(m+1)}$ be given in general positions in an r-space S_r. It is convenient, but not necessary, to let $r=mn+m+n$. We shall assume that the given n-spaces are in an S_{mn+m+n}. Now suppose that these n-spaces are all projectively related, that is, to a given subspace in any one of them corresponds a definite subspace of the same number of dimensions in each of the others. These corresponding subspaces are themselves projectively related.

Now consider a group of corresponding points, one in each of the $m+1$ given n-spaces. These points determine an m-space.

* A *group nucleus* is a neighborhood of the identity; two group nuclei are considered the same if sufficiently small common neighborhoods of the origin are isomorphic.

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