THE PRINCIPAL MATRICES OF A RIEMANN MATRIX*

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1. *Introduction.* A matrix ω with p rows and $2p$ columns of complex elements is called a *Riemann matrix* if there exists a rational $2p$-rowed skew-symmetric matrix C such that

$$(1) \qquad \omega C\omega' = 0, \qquad \pi = i\omega C\bar{\omega}'$$

is positive definite. The matrix C is called *a principal matrix* of ω and it is important in algebraic geometry to know *what are all principal matrices of ω in terms of a given one.* In the present note I shall solve this problem.

2. *Principal Matrices.* A rational $2p$-rowed square matrix A is called a projectivity of ω if

$$(2) \qquad \alpha\omega = \omega A$$

for a p-rowed complex matrix α. The Riemann matrices ω have recently† been completely classified in terms of their projectivities; so we may regard all the projectivities A of ω as known.

A projectivity A is called symmetric if $CA'C^{-1}=A$. Let A be a symmetric projectivity so that if $B=AC$, then $B'=(AC)' = -CA' = -AC = -B$ is a skew-symmetric matrix. Then iAC is Hermitian and so must be

$$(3) \qquad \delta = \omega(iAC)\bar{\omega}' = \alpha(i\omega C\bar{\omega}') = \alpha\pi.$$

Now π is positive definite so that $\pi=\rho\bar{\rho}'$, where ρ is non-singular. Then $\pi^{-1}=(\bar{\rho}')^{-1}\rho^{-1}=\bar{\sigma}'\sigma$ with σ non-singular. Hence $\alpha=\delta\pi^{-1}=\delta\bar{\sigma}'\sigma$ and

$$(4) \qquad \sigma\alpha\sigma^{-1} = \sigma\delta\bar{\sigma}'.$$

The matrix $\sigma\delta\bar{\sigma}'$ is evidently Hermitian and it is well known that then $\sigma\delta\bar{\sigma}'$ and the similar matrix α have *only simple ele-*

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† See my paper *A solution of the principal problem in the theory of Riemann matrices,* Annals of Mathematics, October, 1934.

mentary divisors and *all real characteristic roots*. Thus $\alpha=\beta\gamma\beta^{-1}$, where γ is a real diagonal matrix.

Write

$$\Omega=\begin{pmatrix}\omega\\ \bar{\omega}\end{pmatrix},$$

so that, as is well known, and may easily be computed,

$$A = \Omega^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \bar{\alpha}\end{pmatrix}\Omega = \Lambda\Gamma\Lambda^{-1}, \tag{5}$$

where

$$\Gamma = \begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix}, \qquad \Lambda = \Omega^{-1}\begin{pmatrix}\beta & 0\\ 0 & \bar{\beta}\end{pmatrix}. \tag{6}$$

Then A is similar to the real diagonal matrix Γ and we have proved the following theorem.*

THEOREM 1. *A symmetric projectivity of a Riemann matrix has all simple elementary divisors and all real characteristic roots.*

We may now determine all principal matrices of a given Riemann matrix ω with a given principal matrix C. Let B be a second principal matrix of ω so that $\omega B\omega'=0$. It is well known that $BC=A$ is a projectivity of ω. In fact $\alpha\omega=\omega A$, where $\alpha=\delta\pi^{-1}$ is defined by (3). Moreover $B'=-B$, so that

$$(AC)'=C'A'=-CA'=-AC,$$

and $CA'C^{-1}=A$. Hence $A=BC^{-1}$ is a symmetric projectivity of ω.

The matrix $\delta=i\omega B'\bar{\omega}$ is positive definite if B is a principal matrix of ω. Hence $\sigma\delta'\bar{\sigma}'$ is positive definite and has all positive characteristic roots. The matrices α and γ defined above are similar to $\sigma\alpha\sigma^{-1}=\sigma\delta\bar{\sigma}'$ and have the same characteristic roots, so that the diagonal matrix Γ, whose diagonal elements are these characteristic roots repeated, has all positive diagonal elements. Then A, which is similar to Γ, has all positive characteristic roots.

Conversely, let A be a symmetric projectivity of ω with all positive characteristic roots. Then Γ has all positive diagonal

* The proof by the use of (4) was suggested by certain analogous considerations of N. Jacobson.

elements, α has all positive characteristic roots and so has $\sigma\alpha\sigma^{-1}=\sigma\delta\bar{\sigma}'$. But $\sigma\delta\bar{\sigma}'$ is an Hermitian matrix with characteristic roots all positive. Then $\sigma\delta\bar{\sigma}'$ is positive definite and so is $\delta=i\omega AC\omega'$. Moreover, if $B=AC$, then

$$\omega B\omega'=\omega AC\omega'=\alpha\omega C\omega'=0$$

and B is a principal matrix of ω. We have proved the following result.

THEOREM 2. *Let ω be a Riemann matrix with principal matrix C and let A range over the set of all symmetric projectivities of ω which have positive characteristic roots. Then a rational matrix B is a principal matrix of ω if and only if $B=AC$ with A in the above set.*

3. *Pure Riemann Matrices of the First Kind.* The problem of determining what projectivities of ω are symmetric with all characteristic roots positive is, in general, a complicated one. We may nevertheless solve this problem for the case where ω is a pure Riemann matrix of the first kind.

The multiplication algebra of a pure Riemann matrix is a division algebra D. The centrum of D is a field represented by a field $R(S)$ of all polynomials with rational coefficients of a projectivity S of ω. Algebra D is of the first or second kind according as S is or is not symmetric.

If D is of the first kind, then I have proved* that every projectivity of ω has the form $p(S)$ in $R(S)$ or the form

$$\alpha_1+\alpha_2X+\alpha_3Y+\alpha_4XY, \tag{7}$$

with $\alpha_1, \cdots, \alpha_4$ in $R(S)$, such that

$$YX=-XY,\quad X^2=\xi,\quad Y^2=\eta,\quad (\xi,\eta \text{ in } R(S)). \tag{8}$$

The order of the set of all symmetric projectivities of ω is its singularity index k. If S is symmetric and $R(S)$ has order t, then $k=t$ or $k=3t$ according as we may not or may take both X and Y symmetric, while $k=t$ if D is equivalent to $R(S)$.

Let first $k=t$ so that every symmetric projectivity of ω is in $R(S)$, and let the characteristic roots of S be $\sigma_1, \cdots, \sigma_t$. Then

* Annals of Mathematics, vol. 33 (1932), pp. 311–318.

if $A = p(S)$, the characteristic roots of A are $p(\sigma_j)$ and we have the following theorem.

THEOREM 3. *Let ω be a pure Riemann matrix of the first kind with projectivity algebra D_0 over $R(S)$ having singularity index $k = t$. Then the principal matrices of ω are the matrices*

$$p(S)C, \tag{9}$$

where $p(S)$ is a polynomial in S with rational coefficients such that

$$p(\sigma_j) > 0, \qquad (j = 1, \cdots, t). \tag{10}$$

Next let $k = 3t$ so that every symmetric projectivity of ω has the form

$$A = p_1(S) + p_2(S)X + p_3(S)Y. \tag{11}$$

Then A satisfies the equation in an indeterminate α

$$[\alpha - p_1(S)]^2 = [p_2(S)]^2\xi + [p_3(S)]^2\eta. \tag{12}$$

Hence the characteristic roots of A are the numbers

$$p_1(\sigma_j) \pm \{[p_2(\sigma_j)]^2\xi(\sigma_j) + [p_3(\sigma_j)]^2\eta(\sigma_j)\}^{1/2}.$$

Since X and Y are symmetric we have the well known trivial result

$$\xi(\sigma_j) > 0, \qquad \eta(\sigma_j) > 0. \tag{13}$$

But then the characteristic roots of A are all positive if and only if

$$p_1(\sigma_j) > \{[p_2(\sigma_j)]^2\xi(\sigma_j) + [p_3(\sigma_j)]^2\eta(\sigma_j)\}^{1/2}. \tag{14}$$

We have proved the following theorem.

THEOREM 4. *Let ω be pure with singularity index $k = 3t$ and let $p_1(S)$, $p_2(S)$, $p_3(S)$ be polynomials in S with rational coefficients. Then every principal matrix of ω is given by the set of matrices*

$$[p_1(S) + p_2(S)X + p_3(S)Y]C, \tag{15}$$

with p_1, p_2, p_3 chosen so that (14) *holds.*

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