ON ALGEBRAIC VARIETIES OF k DIMENSIONS IN SPACE OF r DIMENSIONS

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An algebraic k-dimensional variety V_k which is the locus of ∞^k points and not the locus of ∞^{k-h} h-spaces, where $h>0$, possesses numerous characteristics. Certain $2k$ of these will be regarded as essential and all the others may be expressed in terms of them. Besides the order, n, of the variety, we shall define the other $2k-1$ essential characteristics in the following manner.

Consider V_k as belonging to an r-space S_r. A general $(r-k+t)$-space $S_{r-k+t}[1\leqq t\leqq k]$ of S_r intersects V_k in a V_t. The ∞^{2t-1} tangent lines of this V_t form a variety W_{2t} of $2t$ dimensions. Let j_t be the order of W_{2t}, that is, the number of tangent lines of V_t that meet a given $(r-k-t)$-space of S_{r-k+t}. If V_t is projected upon a $(2t-1)$-space of S_{r-k+t}, the projection will have j_t pinch points.

Now an $(r-k+t-1)$-space of S_{r-k+t} intersects the k-dimensional variety V_t in a V_{t-1}. If V_{t-1} possesses a conical point, we say that the $(r-k+t-1)$-space is tangent to V_t. The number of tangent $(r-k+t-1)$-spaces of V_t passing through a given $(r-k+t-2)$-space of S_{r-k+t} is finite. Denote this number by m_t. We say that m_t is the class of V_t. Obviously, $m_1\equiv j_1$.

Thus, we have defined $2k$ of the characteristics of the variety V_k: $n; j_1, j_2, \cdots, j_k; m_1\equiv j_1, m_2, \cdots, m_k$. We regard these as essential.

Our present knowledge of k-dimensional varieties is practically nil, except for the case where $k=2$ and for the case where the varieties are loci of ∞^1 $(k-1)$-spaces. In this note our purpose is to call attention to the fact that, if V_k is the complete intersection of $r-k$ hypersurfaces, $k-1$ of the $2k$ essential characteristics can be expressed in terms of the remaining $k+1$.We find it convenient to express $m_2, m_3, \cdots, m_k$ in terms of n and $j_1, j_2, \cdots, j_k$.

Let the $r-k$ hypersurfaces be of orders $n_1, n_2, \cdots, n_{r-k}$, respectively. Then $n=n_1n_2\cdots n_{r-k}$. By the methods of analytic geometry, we find that the values of the j's and m's are, writing α_i for $n_i - 1$,

$$i_1 = n\sum \alpha_1, \qquad j_2 = n\sum \alpha_1\alpha_2, \quad \cdots, \qquad j_k = n\sum \alpha_1\alpha_2 \cdots \alpha_k;$$

$$m_2 = n(\sum \alpha_1^2 + \sum \alpha_1\alpha_2),$$

$$m_3 = n(\sum \alpha_1^3 + \sum \alpha_1^2\alpha_2 + \sum \alpha_1\alpha_2\alpha_3),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$m_k = n\sum_{\nu} \sum \alpha_1^{\nu_1}\alpha_2^{\nu_2} \cdots \alpha_k^{\nu_k},$$

where

$$\nu = \nu_1 + \nu_2 + \cdots + \nu_k = k.$$

It is not difficult to see that if we eliminate the α's from these equations, we have all the m's expressed in terms of n and the j's. We shall give the values for a few of the m's:

$$nm_2 = j_1^2 - nj_2,$$

$$n^2m_3 = j_1^3 - 2nj_1j_2 + n^2j_3,$$

$$n^3m_4 = j_1^4 - 3nj_1^2j_2 + n^2j_2^2 + 2n^2j_1j_3 - n^3j_4,$$

$$n^4m_5 = j_1^5 - 4nj_1^3j_2 + 3n^2j_1j_2^2 + 3n^2j_1^2j_3 - 2n^3j_2j_3 - 2n^3j_1j_4 + n^4j_5,$$

$$n^5m_6 = j_1^6 - 5nj_1^4j_2 + 4n^2j_1^3j_3 + 6n^2j_1^2j_2^2 - n^3j_2^3 - 6n^3j_1j_2j_3 - 3n^3j_1^2j_4 - 2n^4j_3^2 + 2n^4j_2j_4 + 2n^4j_1j_5 - n^5j_6.$$

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