INDEPENDENT POSTULATES FOR THE "INFORMAL" PART OF PRINCIPIA MATHEMATICA*

BY E. V. HUNTINGTON

1. *Introduction.* It has long been recognized that Section A of Whitehead and Russell's *Principia Mathematica* contains two distinct theories of mathematical logic—one a "formal" or "official" theory, the other an "informal" or "unofficial" theory. The "formal" theory is embodied in a series of numbered propositions, while the "informal" theory includes, besides the numbered propositions, certain other propositions inserted by way of explanation or commentary. Since some of these explanatory propositions are actual additions to the text, not deducible from the numbered propositions, it appears that the "formal" theory is the more restricted (in the number of its theorems) and the "informal" theory the more inclusive of the two.

The contrast between these two theories presents an important problem in the foundations of mathematics; but in spite of the voluminous literature that has grown up around the "formal" theory during the last twenty years, little attention has been paid to the "informal" theory; moreover, the special notation in which the whole of the Principia is expressed is still unfamiliar to many mathematicians.

The purpose of the present paper is to show that the "informal" theory of the Principia, when translated into more familiar mathematical language, is capable of being represented by an ordinary abstract mathematical theory; and for this abstract mathematical theory a set of independent postulates is worked out in the usual way.†

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† The scheme of translation here employed (representing the assertion sign by a "subclass C") was first used in 1933, in my paper entitled *New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's Principia Mathematica*, Transactions of this Society, vol. 35 (1933), pp. 274–304, with corrections on p. 557 and p. 971. The present paper is a simplification and extension of Appendix II of that paper. An earlier scheme of translation (representing the assertion sign by the notation "$=1$") is found in a paper by B. A. Bernstein entitled *Whitehead and Russell's theory of deduction as a mathematical science*, this Bulletin, vol. 37 (1931), pp. 480–488. The "sub-

2. *The Primitive Ideas* (K, C, $+$, $'$). The primitive ideas in the abstract mathematical theory here proposed are four in number:

K = an undefined class of elements, a, b, c, $\cdots$;

C = an undefined subclass within the class K;

$a+b$ = the result of an undefined binary operation on a and b;

a' = the result of an undefined unary operation on a.

It will be assumed without further mention that the subclass C, and hence the main class K, is non-empty.

3. *The Postulates* P1–P8. The postulates which we propose to consider are the following eight.

POSTULATE P1. *If a is in K and b is in K, then $a+b$ is in K.*

POSTULATE P2. *If a is in K, then a' is in K.*

(These two postulates merely ensure that the class K is a "closed set" with respect to the operations $+$ and $'$.)

POSTULATE P3. *If a is in C, then a is in K.*

(This postulate merely ensures that C is a subclass in K.)

In the following postulates, a and b are assumed to be elements of K.

POSTULATE P4. *If $a+b$ is in C, then $b+a$ is in C.*

POSTULATE P5. *If a is in C, then $a+b$ is in C.*

DEFINITION. The notation (a is in C') shall mean (a is in K and a is not in C).

POSTULATE P6. *If a is in K and a' is in C, then a is in C'.*

POSTULATE P7. *If a is in K and a' is in C', then a is in C.*

POSTULATE P8. *If $a+b$ is in C and a' is in C, then b is in C.*

Any system (K, C, $+$, $'$) which satisfies these Postulates P1–P8 may be called an *informal Principia system*, for reasons which will be explained in a later section; and the independence of these eight postulates will be established in the usual way.

Three supplementary postulates, P9–P11, will be introduced in the following paper.

class C" is believed to be preferable to the "$=1$," since the notation "$=1$" appears to suggest that all true propositions are in some sense "equal" to a single (selected) proposition, whereas the "subclass C" does not suggest any such artificial restriction.

4. *Immediate Consequences of Postulates* P1–P8. The following theorems are immediate consequences of Postulates P1–P8, no matter what interpretation may be given to the undefined symbols (K, C, $+$, $'$).

P12. *If a is in C', then a' is in C.*

PROOF. By definition, a is in K and a is not in C. Hence by P2, a' is in K. Suppose a' is not in C. Then by definition, a' is in C', whence, by P7, a is in C, contrary to hypothesis. Hence a' is in C. (This Theorem P12 is the converse of P6.)

P13. *If a is in C, then a' is in C'.*

PROOF. By P3, a is in K, whence by P2, a' is in K. Suppose a' is in C. Then by P6, a is in C', whence by definition, a is not in C, contrary to hypothesis. Hence a' is not in C. Hence by definition, a' is in C'. (P13 is the converse of P7.)

P14. *If a is in K and a is not in C', then a is in C.*

PROOF. Suppose a is not in C. Then by definition and the first part of the hypothesis, a is in C', contrary to the second part of the hypothesis. Hence a is in C.

COROLLARY 1. The class K is divided into two non-empty, mutually exclusive subclasses, C and C', which together exhaust the class K. (Simple dichotomy.)

To prove that there exists at least one element in C', note that, by tacit assumption, there exists at least one element a in C; then this element a determines an element a' which by P13 is in C'. That C and C' are mutually exclusive and together exhaust the class K follows from the definition of C'.

P15. *If $a+b$ is in C', then $b+a$ is in C'.*

PROOF. By P1, $b+a$ is in K. Suppose $b+a$ is not in C'. Then by P14, $b+a$ is in C, whence by P4, $a+b$ is in C, contrary to hypothesis. Hence $b+a$ is in C'.

P16. *If $a+b$ is in C and a is in C', then b is in C.*

PROOF. By P12, a' is in C. Hence, by P8, b is in C.

COROLLARY 2. If $a+b$ is in C, then at least one of the elements a and b is in C.

P17. *If $a+b$ is in C', then a is in C' and b is in C'.*

PROOF. Suppose a is not in C'. Then by P14, a is in C, whence by P5, $a+b$ is in C, contrary to hypothesis. Hence a is in C'. Again, suppose b is not in C'. Then by P14, b is in C, whence by P5, $b+a$ is in C, whence by P4, $a+b$ is in C, contrary to hypothesis. Hence b is in C'.

P18. *If a is in C' and b is in C', then $a+b$ is in C'.*

PROOF. Suppose $a+b$ is not in C'. Then by P14, $a+b$ is in C, whence by P16 and the first part of hypothesis, b is in C, contrary to the second part of hypothesis. Hence $a+b$ is in C'.

P19. (1) *If a is in C, then $(a')'$ is in C; and conversely.*
(2) *If $(a')'$ is in C, then a is in C.*
(3) *If a is in C', then $(a')'$ is in C'; and conversely.*
(4) *If $(a')'$ is in C', then a is in C'.*

PROOF. (1) By P13, a' is in C', whence by P12, a'' is in C.
(2) By P6, a' is in C', whence by P7, a is in C.
(3) By P12, a' is in C, whence by P13, a'' is in C'.
(4) By P7, a' is in C, whence by P6, a is in C'.

5. *Deduction of* P20–P29 *from* P1–P8. The following theorems P20–P29 are further consequences of Postulates P1–P8. (The references in brackets are to the Principia, as explained in the following section.)

P20. *If a is in C, then a is in K.* [Page 92(3)]
P21. *If a is in C and $a'+b$ is in C, then b is in C.* [*1.1]
P22. *If a is in K, then $(a+a)'+a$ is in C.* [*1.2]
P23. *If a, b, etc. are in K, then $b'+(a+b)$ is in C.* [*1.3]
P24. *If a, b, etc. are in K, then $(a+b)'+(b+a)$ is in C.* [*1.4]
P25. *If a, b, c, etc. are in K, then*
$(b'+c)'+[(a+b)'+(a+c)]$ is in C. [*1.6]
P26. *If a is in K and b is in K, then $a+b$ is in K.* [*1.71]
P27. *If a is in K, then a' is in K.* [*1.7]
P28. *If a' is in C, then a is not in C.* [Page 93(5)]
P29. *If $a+b$ is in C, then at least one of the elements*
a and b is in C. [Page 93(6)]

Of these theorems, P20, P26, P27, P28, P29 follow at once from P3, P1, P2, P6, P16. The proofs of the remaining theorems are as follows.*

* For valuable suggestions in this connection I am indebted to Alonzo Church and K. E. Rosinger.

PROOF OF P21. From a in C follows a'' in C (by P13 and P12). Hence we have a'' in C and $a'+b$ in C, whence by P8, b is in C.

PROOF OF P22. If a is in C, then by P5 and P4, $(a+a)'+a$ is in C. If a is not in C, then a is in C', whence by P18, $a+a$ is in C', whence by P12, $(a+a)'$ is in C, whence by P5, $(a+a)'+a$ is in C.

PROOF OF P23. If b is in C, then by P5 and P4, $a+b$ is in C, whence by P5 and P4, $b'+(a+b)$ is in C. If b is not in C, then b is in C', whence by P12, b' is in C, whence by P5, $b'+(a+b)$ is in C.

PROOF OF P24. Suppose $(a+b)'+(b+a)$ is not in C. Then $(a+b)'+(b+a)$ is in C', whence by P17, $(a+b)'$ is in C', whence by P7, $a+b$ is in C, whence by P4, $b+a$ is in C, whence by P5 and P4, $(a+b)'+(b+a)$ is in C.

PROOF OF P25. Suppose $(b'+c)'+[(a+b)'+(a+c)]$ is in C'. Then by P17, $(a+b)'+(a+c)$ is in C', whence by P17, $a+c$ is in C', whence by P17, a is in C' and c is in C'. By P12, a' is in C. Also, by P17, $(a+b)'$ is in C', whence by P7, $a+b$ is in C, whence by P8, b is in C, whence by P13, b' is in C'. Then by P18, $b'+c$ is in C', whence by P12, $(b'+c)'$ is in C, whence by P5, $(b'+c)'+[(a+b)'+(a+c)]$ is in C.

6. *Correspondence between* P20–P29 *and the "Informal" Principia*. The validity of Theorems P20–29 of course does not depend on any particular interpretation of the undefined symbols (K, C, $\vdash$, $'$). But if the undefined class K is interpreted as the class of entities called "elementary propositions" in the Principia; and if the undefined subclass C is interpreted as the class of "true" elementary propositions, distinguished from other propositions in the Principia by the use of the assertion sign, $\vdash$; and if the undefined element $a+b$ is interpreted as the proposition "a or b," denoted in the Principia by $a \vee b$ (conveniently read "a wedge b"); and if the undefined element a' is interpreted as the proposition "not-a," denoted in the Principia by $\sim a$ (conveniently read "curl a"); then these propositions P20–P29 will be found to correspond precisely to the "primitive propositions" on which the "informal" theory of the Principia is based.*

* The words "wedge" and "curl" (like the word "horse-shoe") are due to H. M. Sheffer.

Thus, Theorem P20 corresponds to the fact that in the Principia a "true proposition" is at any rate a "proposition"; Theorems P21–P27 correspond to the "formal" primitive propositions in the Principia, as indicated by the starred numbers in brackets; Theorem P28 corresponds to the Principia's "informal" explanation of the meaning of $\sim a$; and Theorem P29 corresponds to the Principia's "informal" explanation of the meaning of $a \vee b$.

Hence our deduction of Theorems P20–P29 from the Postulates P1–P8 is in effect a deduction of the whole "informal" system of the Principia from these Postulates P1–P8.*

7. *Deduction of* P1–P8 *from* P20–P29. We now show, conversely, that P1–P8 can be deduced from P20–P29, no matter

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* In detail, the steps of the translation may be justified as follows.

The translation of the assertion sign, "$\vdash$," into the "subclass C" is suggested by the following statement on page 92 of the Principia (vol. I, second edition, 1925): "The sign '$\vdash$' may be read 'it is true that';" thus "$\vdash : \sim p \vee q$" means "it is true that either p is false or q is true."

The theorems P28 and P29 are translations of the following passages on page 93 of the Principia: "The proposition '$\sim p$' means 'not-p' or 'p is false';" and "The proposition '$p \vee q$' means 'either p is true or q is true,' where the alternatives are not to be mutually exclusive."

The translation of *1.2, *1.3, *1.4, *1.6 into P22–P25 is immediate, in view of the definition of $p \supset q$ in *1.01, namely, $p \supset q \cdot = \cdot \sim p \vee q$.

The translation of *1.71 and *1.7 into P26 and P27 is also immediate.

In regard to *1.1, which reads in the original: "Anything implied by a true elementary proposition is true," the authors of the Principia state that they "cannot express this principle symbolically," and that it is not the same as "if p is true, then if p implies q, q is true." But in view of the Principia's definition of "p implies q," namely, $\sim p \vee q$ (*1.01), and in view of the actual use which is made of *1.1 throughout the Principia (in which connection *8.12 is of interest), it appears that for our present purposes *1.1 may fairly be translated into P21.

The primitive proposition *1.5 (the associative law for $\vee$) is omitted because it is now known to be a consequence of the other numbered primitive propositions (proof by Bernays in 1926, reproduced in Transactions of this Society, vol. 35 (1933), p. 292).

The primitive propositions *1.11 and *1.72, together with the primitive idea "assertion of a propositional function," are omitted in accordance with instructions found in the Introduction to the second edition of the Principia, page xiii.

Thus all the primitive ideas and primitive propositions of the "informal" Principia are accounted for.

what interpretation may be given to the undefined symbols (K, C, $+$, $'$).

P1, P2, P3.

These propositions follow immediately from P26, P27, P20.

P4. If $a+b$ is in C, then $b+a$ is in C.

PROOF. By hypothesis, $a+b$ is in C. By P24, $(a+b)'+(b+a)$ is in C. Hence by P21, $b+a$ is in C.

P5. If a is in C, then $a+b$ is in C.

PROOF. By P23, $a'+(b+a)$ is in C. Hence by P21, $b+a$ is in C, whence by P4, $a+b$ is in C.

DEFINITION. (a is in C') means (a is in K and a is not in C).

LEMMA. If a is in K, then $a'+a$ is in C.

PROOF. Case 1. Suppose a' is in C. Then by P5, $a'+a$ is in C. Case 2. Suppose a' is not in C. By P23, $a'+(a'+a)$ is in C. Hence by P29, $a'+a$ is in C.

ALTERNATIVE PROOF (without using P29). By P25, $[(a+a)'+a]'+\{[a'+(a+a)]'+(a'+a)\}$ is in C. But by P22, $(a+a)'+a$ is in C. Hence by P21, $[a'+(a+a)]'+(a'+a)$ is in C. But by P23, $a'+(a+a)$ is in C. Hence by P21, $a'+a$ is in C.

P6. If a is in K and a' is in C, then a is in C'.

PROOF. By P28 and the definition of C'.

P7. If a is in K and a' is in C', then a is in C.

PROOF. By lemma, $a'+a$ is in C. But by definition, a' is not in C. Hence by P29, a is in C.

P8. If $a+b$ is in C and a' is in C, then b is in C.

PROOF. By P28, a is not in C. Hence by P29, b is in C.

We note in passing that P22 and P25 are deducible from the other propositions of the list P20–P29.

8. *Equivalence of the "Informal" Principia and Postulates* P1–P8. The propositions P20–P29 have been deduced from P1–P8; and conversely, P1–P8 have been deduced from P20–P29; so that the set of propositions P1–P8 is equivalent to the set of propositions P20–P29. But P20–P29 have been shown to correspond to the primitive propositions of the "informal" Principia. Hence, if our scheme of translation is adequate, the Postulates P1–P8 fairly represent the whole "informal" theory of the Principia.

9. *Independence of Postulates* P1–P8. The independence of Postulates P1–P8 is established by the following examples of systems (K, C, $+$, $'$), each of which violates the like-numbered postulate and satisfies all the others.

EXAMPLE P1.

K = a class of four elements, represented by the "tags" 1, 2, 3, 4;
$C=1, 2$;
$a+b$ and a' as in the table (w being any number not in the class K).
Here P1 clearly fails. All the other postulates are satisfied.

$+$	1	2	3	4	$'$
1	1	1	1	1	4
2	1	2	1	2	3
3	1	1	3	w	2
4	1	2	w	4	1

EXAMPLE P2.

$K=1, 2, 3, 4$;
$C=1, 2$;
$a+b$ and a' as in the table.
Here P2 fails since $1'$ is not in K.
All the other postulates are satisfied.

$+$	1	2	3	4	$'$
1	1	1	1	1	w
2	1	2	1	2	3
3	1	1	3	3	2
4	1	2	3	4	1

EXAMPLE P3.

$K=1, 2$;
$C=1, 3$;
$a+b$ and a' as in the table.
Here P3 fails on $a=3$.
All the other postulates are found to be satisfied (the subclass C' comprising the single element 2).

$+$	1	2	3	$'$
1	1	1	1	2
2	1	2	w	1
3	1	w	3	3

EXAMPLE P4.

$K=1, 2, 3, 4$;
$C=1, 2$;
$a+b$ and a' as in the table.
Here P4 fails, since $1+3(=1)$ is in C and $3+1$ $(=3)$ is not. The other seven postulates are all satisfied. (Postulate P8 is satisfied "vacuously," since the condition $a+b$ in C and a' in C does not occur in this system.) Note. This system does not satisfy the commutative law $a+b=b+a$.

$+$	1	2	3	4	$'$
1	1	1	1	1	4
2	2	2	2	2	3
3	3	3	3	3	2
4	4	4	4	4	1

EXAMPLE P5.

$K=1, 2, 3, 4$;
$C=1, 2$;
$a+b$ and a' as in the table
Here P5 fails, since 2 is in C and $2+3$ is not. All the other postulates are found to satisfied (P8 vacuously).

+	1	2	3	4	′
1	1	2	3	4	4
2	2	2	4	4	3
3	3	4	3	4	2
4	4	4	4	4	1

EXAMPLE P6.

$K=1, 2, 3, 4$;
$C=1, 2, 3, 4$;
$a+b$ and a' as in the table. Here C' is empty.
Postulate P6 fails, since $2'$ $(=3)$ is in C and 2 is not in C'. All the other postulates are satisfied (P7 vacuously).

+	1	2	3	4	′
1	1	1	1	1	4
2	1	2	1	2	3
3	1	1	3	3	2
4	1	2	3	4	1

EXAMPLE P7.

$K=1, 2, 3, 4$;
$C=1$;
$a+b$ and a' the same as in the table for Example P6. Here $C'=2, 3, 4$. Postulate P7 fails, since $2'$ $(=3)$ is in C' and 2 is not in C. All the other postulates are satisfied.

EXAMPLE P8.

$K=1, 2, 3, 4, 5, 6, 7, 8$;
$C=1, 2, 3, 4$;
$a+b$ and a' as in the table.
Here P8 fails, since $5+6$ is in C and $5'$ is in C but 6 is not in C. All the other postulates are satisfied.

+	1	2	3	4	5	6	7	8	′
1	1	1	1	1	1	1	1	1	8
2	1	2	1	1	2	2	1	2	7
3	1	1	3	1	3	1	3	3	6
4	1	1	1	4	1	4	4	4	5
5	1	2	3	1	5	2	3	5	4
6	1	2	1	4	2	6	4	6	3
7	1	1	3	4	3	4	7	7	2
8	1	2	3	4	5	6	7	8	1

These eight examples show that no one of the Postulates P1–P8 can be deduced from the other seven.

10. *Examples of Systems* (K, C, $+$, $'$) *Satisfying Postulates* P1–P8. The *consistency* of the Postulates P1–P8 is established

by the existence of any one of the following examples of systems (K, C, $+$, $'$), each of which satisfies all the postulates.

EXAMPLE 0.1.

K = the class comprising the two numbers 1 and 2;
C = the class comprising the single number 1;
$a+b$ and a' = the numbers given by the adjoining table.

+	1	2	′
1	1	1	2
2	1	2	1

EXAMPLE 0.2.

K = the four numbers 1, 2, 3, 4;
C = the two numbers 1, 2;
$a+b$ and a' = the numbers given by the table.

+	1	2	3	4	′
1	1	1	1	1	4
2	1	2	1	2	3
3	1	1	3	3	2
4	1	2	3	4	1

EXAMPLE 0.3.

K = 1, 2, 3, 4, 5, 6, 7, 8;
C = 1, 2, 3, 4;
$a+b$ and a' as given by the table.

+	1	2	3	4	5	6	7	8	′
1	1	1	1	1	1	1	1	1	8
2	1	2	1	2	1	2	1	2	7
3	1	1	3	3	1	1	3	3	6
4	1	2	3	4	1	2	3	4	5
5	1	1	1	1	5	5	5	5	4
6	1	2	1	2	5	6	5	6	3
7	1	1	3	3	5	5	7	7	2
8	1	2	3	4	5	6	7	8	1

Other examples will be found in the following supplementary paper.

HARVARD UNIVERSITY