This determinant will vanish only when the points on the curve C corresponding to the values a_1, a_2, and a_3 lie on a straight line. When the determinant vanishes, we have $f(y) = f(y')$, and hence the points P and P' coincide.

If C is a straight line, the determinant vanishes identically and all curves have the closure property. If C is not cut by any straight line in more than two points, then none of the curves have the closure property.

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NOTE ON HOMOGENEOUS FUNCTIONALS*

BY L. S. KENNISON

The classical formula of Euler for functions homogeneous in n variables is as follows.

Let $f(x_1, \cdots, x_n)$ *be a differentiable function of the* n *variables,* $x_1, \cdots, x_n$, *such that*

$$f(\lambda x_1, \cdots, \lambda x_n) = \lambda^p f(x_1, \cdots, x_n). \tag{1}$$

Then we have

$$x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} = p f(x_1, \cdots, x_n). \tag{2}$$

The following analog of this formula for functionals of one variable was proved by E. Freda.†

Let $F \,|\, [f(x)] \,|$ *be a functional with a Fréchet differential* $\delta F = \int_0^1 F' \,|\, [f(x)];\ \xi \,|\, \delta f(\xi) d\xi + \sum_1^n A_s \,|\, [f(x)] \,|\, \delta f(x_s)$, *where* $x_1, \cdots, x_n$ *are points of the interval* $(0, 1)$, *and such that*

$$F \,|\, [\lambda f(x)] \,| = \lambda^r F \,|\, [f(x)] \,|.$$

Then

$$\left\{ \frac{\partial}{\partial \lambda} F \,|\, [f(x)(1 + \lambda)] \,| \right\}_{\lambda=0} = r F \,|\, [f(x)] \,|.$$

Theorem 2 of this paper will be a generalization of this theorem of Freda.

The following theorem is classical.

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* Presented to the Society, January 19, 1932.

† Rendiconti dei Lincei, (5), vol. 24 (1915), p. 1035.

Let $f(x_1, \cdots, x_n)$ *be a differentiable function of the n variables* $x_1, \cdots, x_n$, *such that*

$$f(\lambda x_1, \cdots, \lambda x_n) = K(\lambda) f(x_1, \cdots, x_n) ;$$

then $K(\lambda)$ *is a power of* λ.

In Theorem 1 we generalize this theorem to functionals.

DEFINITION 1. *Let* $Y_1, \cdots, Y_n$ *be n functions each of an arbitrary number (including possibly zero) of variables. If for every determination of the set* $Y_1, \cdots, Y_n$, *(on a certain range), there is determined a function F of a definite number of variables*, $z_1, \cdots, z_m$, *then we say that F is a functional of the n functions* Y_i *and write*

$$F(z_1, \cdots, z_m) = F[Y_1, \cdots, Y_n, z_1, \cdots, z_m].$$

The distinction between F as depending on a Y which is a function of no variables (an independent variable), and on the z's, is largely arbitrary, being a question of what ranges our independent and dependent variables shall have. That is, the dependent variable is not the number F, corresponding to the determination of the set $(Y_1, \cdots, Y_n, z_1, \cdots, z_m)$, but the function $F(z_1, \cdots, z_m)$ corresponding to the determination of the set $(Y_1, \cdots, Y_n)$.

Frequently it will be convenient to consider a normed vector space each of whose elements is a suitable determination of a set of functions $(Y_1, \cdots, Y_n)$. We shall denote the norm of this element by $\| Y \|$. If the functional $F(z_1, \cdots, z_m)$ be defined over a portion of such a space, then we may define the Fréchet differential of F, when it exists, as follows. (From this point on we shall omit writing the z's.)

DEFINITION 2. *Let* $F[Y_1, \cdots, Y_n]$ *be a functional defined over a portion of a normed vector space of elements* $(Y_1, \cdots, Y_n)$. *Let us consider an element of this space* $(\Delta Y_1, \cdots, \Delta Y_n)$. *Define* $\Delta F[Y_1, \cdots, Y_n; \Delta Y_1, \cdots, \Delta Y_n]$ *to be* $F[Y_1+\Delta Y_1, \cdots, Y_n+\Delta Y_n]-F[Y_1, \cdots, Y_n]$. *Let* $\eta=\|\Delta Y\|$. *Then the Fréchet differential*, $dF[Y_1, \cdots, Y_n; \Delta Y_1, \cdots, \Delta Y_n]$ *is defined to have the following properties*:

(A) *dF is linear and homogeneous* in the* ΔY*'s.*

* This means that $dF[Y_1, \cdots, Y_n; \lambda_1\Delta Y_1+\mu_1\Delta' Y_1, \cdots, \lambda_n\Delta Y_n+\mu_n\Delta' Y_n] = A_1\lambda_1+B_1\mu_1+ \cdots +A_n\lambda_n+B_n\mu_n$, where the λ's and μ's are independent variables. My attention has been called to the fact that this definition differs from Fréchet's in that he uses linear in Property (A) to include distributive and continuous. The continuity is not needed for this paper.

(B) $(\Delta F - dF)/\eta$ *approaches zero with* η.

The only further property of the Fréchet differential needed below is that

$$\frac{d}{d\lambda} F[\lambda Y_1, \cdots, \lambda Y_n] = dF[\lambda Y; Y].$$

This follows from an easy generalization of Gateaux's formula* provided $\|\lambda Y\| = |\lambda| \cdot \|Y\|$. Hence we shall not need to specify the particular normed vector space we are using.

THEOREM 1. *Let* $F[Y_1, \cdots, Y_n]$ *denote a functional of the* n *functions* $Y_1, \cdots, Y_n$ *which possesses a Fréchet differential* $dF[Y_1, \cdots, Y_n; \Delta Y_1, \cdots, \Delta Y_n]$, *and such that*

$$F[\lambda Y_1, \cdots, \lambda Y_n] = K(\lambda) F[Y_1, \cdots, Y_n]. \tag{3}$$

Then $K(\lambda)$ *is a power of* λ, *say* λ^p.

THEOREM 2. *Under the conditions of Theorem* 1,

$$dF[Y_1, \cdots, Y_n; Y_1, \cdots, Y_n] = pF[Y_1, \cdots, Y_n], \tag{4}$$

where p *is the exponent of* λ *in the conclusion of that theorem.*

These theorems will be proved together. Differentiating (3) with respect to λ, we have

$$dF[\lambda Y; Y] = K'(\lambda) F[Y]. \tag{5}$$

Taking the Fréchet differential of (3), we find

$$\lambda dF[\lambda Y; \Delta Y] = K(\lambda) dF[Y; \Delta Y]. \tag{6}$$

Letting $\Delta Y = Y$ in (6), and eliminating $dF[\lambda Y; Y]$ between (5) and (6), we obtain

$$\frac{\lambda K'(\lambda)}{K(\lambda)} = \frac{dF[Y; Y]}{F[Y]}. \tag{7}$$

The left side is independent of the Y's and the right of λ. Hence each is independent of both, say equal to p. Integrating the differential equation for $K(\lambda)$ obtained by setting the left side equal to p, we have

$$K(\lambda) = C\lambda^p.$$

* Paul Lévy, *Leçons d'Analyse Fonctionnelle*, p. 51.

Letting $\lambda=1$ in (3) we have $C=1$, and Theorem 1 is proved. Letting the right side of (7) equal p gives the conclusion of Theorem 2.

Formula (4) is the analog for functionals of Euler's formula above, and is proved under no restrictions as to the form of the Fréchet differential. We shall now prove the following converse of Theorem 2.

THEOREM 3. *If $F[Y_1, \cdots, Y_n]$ denotes a functional of the n functions $Y_1, \cdots, Y_n$ which has a Fréchet differential $dF[Y_1, \cdots, Y_n; \Delta Y_1, \cdots, \Delta Y_n]$, and if* (4) *holds, then* (3) *also holds, with $K(\lambda)=\lambda^p$.*

Let F be any such functional satisfying (4), and let $R[Y, \lambda] = F[\lambda Y]-\lambda^p F[Y]$. We have

$$R[Y, 1] \equiv 0. \tag{8}$$

Moreover,

$$\frac{dR}{d\lambda} = dF[\lambda Y; Y] - p\lambda^{p-1}F[Y].$$

By (4), we have

$$pF[\lambda Y] = dF[\lambda Y; \lambda Y] = \lambda dF[\lambda Y; Y];$$

hence

$$\frac{dR}{d\lambda} = \frac{p}{\lambda}F[\lambda Y] - p\lambda^{p-1}F[Y] = \frac{p}{\lambda}R[Y; \lambda].$$

Integrating this differential equation, we obtain

$$R[Y; \lambda] = C[Y]\lambda^p. \tag{9}$$

We write our constant of integration as $C[Y]$ as it may be assumed independent of λ only. Letting $\lambda=1$, and using (8) we have $C[Y]=0$. Hence $R[Y; \lambda] = F[\lambda Y]-\lambda^p F[Y]=0$.

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