A NOTE ON A CERTAIN PROPERTY OF A FAMILY OF CURVES

BY ALBERT WERTHEIMER

1. *Introduction.* In studying methods of constructing alignment charts for sets of empirical curves, it was found necessary to consider a certain property of the curves which we will call the closure property. Let C_1, C_2, and C_3 be three plane curves such that C_2 lies between C_1 and C_3; take any point P on C_2 and make the following sequence of projections. Project P vertically on C_3 into P_3, project P_3 horizontally on C_1 into P_1, project P_1 vertically on C_2 into P_2, project P_2 horizontally on C_3 into P_3', project P_3' vertically on C_1 into P_1', finally project P_1' on C_2 into P'. If the points P and P' coincide for all points on C_2, the three curves are said to have the closure property.

2. *Curves with the Closure Property.* Now consider the one-parameter family of curves given by

$$f(y) + g(a)h(x) + k(a) = 0, \tag{1}$$

defined in the region $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$, $m \leqq a \leqq n$, where the functions f, g, h, and k are continuous and single-valued, and let a curve C be defined by the equations

$$x = g(a), \qquad y = k(a), \qquad (m \leqq a \leqq n).$$

Then we have the following result.

THEOREM. *Those sets of three curves of* (1), *and only those, which correspond to values of a at which a straight line cuts the curve C, have the closure property.*

PROOF. Consider three curves C_1, C_2, and C_3 corresponding respectively to the parametric values a_1, a_2, and a_3. Now take any point $P(x, y)$ on C_2 and project it into $P'(x, y)$ as described above. Making use of (1), we get

$$f(y) - f(y') = -\frac{1}{g(a_3)} \begin{vmatrix} g(a_1) & k(a_1) & 1 \\ g(a_2) & k(a_2) & 1 \\ g(a_3) & k(a_3) & 1 \end{vmatrix}.$$

This determinant will vanish only when the points on the curve C corresponding to the values a_1, a_2, and a_3 lie on a straight line. When the determinant vanishes, we have $f(y)=f(y')$, and hence the points P and P' coincide.

If C is a straight line, the determinant vanishes identically and all curves have the closure property. If C is not cut by any straight line in more than two points, then none of the curves have the closure property.

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NOTE ON HOMOGENEOUS FUNCTIONALS*

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The classical formula of Euler for functions homogeneous in n variables is as follows.

Let $f(x_1, \cdots, x_n)$ *be a differentiable function of the* n *variables,* $x_1, \cdots, x_n$, *such that*

$$(1) \qquad f(\lambda x_1, \cdots, \lambda x_n) = \lambda^p f(x_1, \cdots, x_n).$$

Then we have

$$(2) \qquad x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} = p f(x_1, \cdots, x_n).$$

The following analog of this formula for functionals of one variable was proved by E. Freda.†

Let $F|[f(x)]|$ *be a functional with a Fréchet differential* $\delta F = \int_0^1 F' \mid [f(x)];\ \xi | \delta f(\xi) d\xi + \sum_1^n A_s | [f(x)] | \delta f(x_s)$, *where* $x_1, \cdots, x_n$ *are points of the interval* $(0, 1)$, *and such that*

$$F \mid [\lambda f(x)] \mid = \lambda^r F \mid [f(x)] \mid .$$

Then

$$\left\{ \frac{\partial}{\partial \lambda} F \mid [f(x)(1+\lambda)] \mid \right\}_{\lambda=0} = rF \mid [f(x)] \mid .$$

Theorem 2 of this paper will be a generalization of this theorem of Freda.

The following theorem is classical.

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* Presented to the Society, January 19, 1932.

† Rendiconti dei Lincei, (5), vol. 24 (1915), p. 1035.