which, by (16), may be written in the form

$$\sum_{\rho=0}^{k}(-1)^{\rho}\binom{\rho}{k}x^{\rho}\,\Gamma_{n+k-\rho}, \tag{23}$$

and hence is expressible in terms of the Γ_n directly.

These results would be useful in determining the systems of Appell polynomials generated by a general doubly periodic function of the second kind.

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ON DOUBLE RIEMANN-STIELTJES INTEGRALS*

BY J. A. CLARKSON

1. *Introduction.* A recent study by Clarkson and Adams† of functions $f(x, y)$ of bounded variation naturally leads one to the consideration of double Stieltjes integrals. The present paper is devoted to the discussion of certain questions concerning such integrals.

Stieltjes defined the symbol

$$\int_a^b f(x)d\phi(x) \tag{1}$$

by means of the sum

$$\sum_{i=1}^{n} f(\xi_i)[\phi(x_i) - \phi(x_{i-1})],$$

$$(a = x_0 < x_1 < x_2 < \cdots < x_n = b,\ x_{i-1} \leqq \xi_i \leqq x_i).$$

If this sum approaches a finite limit when the norm of the subdivisions approaches zero, (1) is defined as this limit; otherwise (1) is not defined. He showed that for a given $\phi(x)$, a sufficient condition that (1) should exist for every continuous function

* Presented to the Society, December 27, 1933.

† J. A. Clarkson and C. R. Adams, *On definitions of bounded variation for functions of two variables*, Transactions of this Society, vol. 35 (1933), pp. 824–854.

$f(x)$ is that $\phi(x)$ be of bounded variation. Pollard† has shown that this condition is also necessary.

We consider two separate extensions of this notion to functions of two variables. The first is due to Fréchet,‡ who gave the following definition. Assume that $f(x, y)$ and $\phi(x, y)$ are defined over the rectangle $R(a \leqq x \leqq b,\ c \leqq y \leqq d)$; let R be divided into rectangular subdivisions, or *cells*, by the *net* of straight lines $x = x_i$, $y = y_j$, $(a = x_0 < x_1 < x_2 < \cdots < x_m = b$, $c = y_0 < y_1 < y_2 < \cdots < y_n = d)$; let ξ_i, η_j be any numbers satisfying the inequalities

$$x_{i-1} \leqq \xi_i \leqq x_i, \quad y_{j-1} \leqq \eta_j \leqq y_j, \qquad (i = 1, 2, 3, \cdots, m;\ j = 1, 2, 3, \cdots, n);$$

and for all i, j let

$$\Delta_{11}\phi(x_i, y_j) = \phi(x_{i-1}, y_{j-1}) - \phi(x_{i-1}, y_j) - \phi(x_i, y_{j-1}) + \phi(x_i, y_j).$$

Then if the sum

$$S = \sum_{i,j=1}^{m,n} f(\xi_i, \eta_j)\Delta_{11}\phi(x_i, y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to ϕ is said to exist. We call this limit the *restricted integral*, and designate it by the symbol

$$\int_a^b \int_c^d f(x, y)d_x d_y \phi(x, y). \tag{2}$$

If in the above formulation S is replaced by the sum

$$S^* = \sum_{i,j=1}^{m,n} f(\xi_{ij}, \eta_{ij})\Delta_{11}\phi(x_i, y_j),$$

where ξ_{ij}, η_{ij} are any numbers satisfying the inequalities

$$x_{i-1} \leqq \xi_{ij} \leqq x_i,\ y_{j-1} \leqq \eta_{ij} \leqq y_j, \qquad (i = 1, 2, 3, \cdots, m;\ j = 1, 2, 3, \cdots, n),$$

† S. Pollard, *The Stieltjes integral and its generalizations*, Quarterly Journal of Mathematics, vol. 49 (1920), pp. 73–138.

‡ Fréchet, *Extension au cas des intégrals multiples d'une définition de l'intégrale due à Stieltjes*, Nouvelles Annales de Mathématiques, (4), vol. 10 (1910), pp. 241–256.

we call the limit, when it exists, the *unrestricted integral*,† and designate it by the symbol

$$(3) \qquad (*)\int_a^b \int_c^d f(x, y) d_x d_y \phi(x, y).$$

Clearly the existence of (3) implies both the existence of (2) and its equality to (3); on the other hand the existence of (2) does not imply the existence of (3), as we shall presently see.

Fréchet‡ has shown that a sufficient condition for the existence of the restricted integral (2) for every continuous integrand is that the integrator function $\phi(x, y)$ be of bounded variation in the sense of Vitali§ (we write $\phi \subset V$). Of special interest in connection with bilinear functionals is the case in which the integrand is *factorable*; that is, $f(x, y) = g(x)h(y)$. In this case Fréchet|| has shown that a sufficient condition for the existence of the restricted integral (2) for every factorable integrand with continuous factors is that $\phi(x, y)$ be of bounded variation in a certain sense of his own|| ($\phi \subset F$). The condition of Fréchet is known to be weaker¶ than that of Vitali.

The following questions naturally present themselves:

(a) In each case is the sufficient condition of Fréchet also necessary?

(b) In each case is the condition shown by Fréchet to be suffi-

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† Such integrals have recently been considered or employed by T. H. Hildebrandt and I. J. Schoenberg, *On linear functional operations* . . . , Annals of Mathematics, vol. 34 (1933), pp. 317–328; I. J. Schoenberg, *On finite-rowed systems of linear inequalities* . . . , II, Transactions of this Society, vol. 35 (1933), pp. 452–478; and C. R. Adams, *Hausdorff transformations for double sequences*, this Bulletin, vol. 39 (1933), pp. 303–312.

‡ Fréchet, loc. cit.

§ The function $\phi(x, y)$ is said to be of bounded variation in the Vitali sense if there exists a positive quantity M such that for every net on R we have $\sum_{i,j=1}^{m,n} |\Delta_{11}\phi(x_i, y_j)| < M$; and to be of bounded variation in the Fréchet sense if there exists a positive quantity M such that we have $\sum_{i,j=1}^{m,n} \epsilon_i \bar{\epsilon}_j \Delta_{11}\phi(x_i, y_j) < M$ for every net on R and every possible choice of the numbers ϵ_i and $\bar{\epsilon}_j$ equal to $+1$ or -1.

|| Fréchet, *Sur les fonctionnelles bilinéaires*, Transactions of this Society, vol. 16 (1915), pp. 215–234.

¶ See Littlewood, *On bounded bilinear forms in an infinite number of variables*, Quarterly Journal of Mathematics, Oxford Series, vol. 1 (1930), pp. 164–174, or Clarkson and Adams, loc. cit.

cient for the existence of the restricted integral also sufficient for the existence of the unrestricted integral?

(c) In the second case is the condition of Fréchet, employed when the integrand is factorable, sufficient for the existence of the integral (either restricted or unrestricted) when the integrand is continuous but not necessarily factorable?

We shall show (i) that the answer to both questions (a) is affirmative, which implies a negative answer to (c); and (ii) that the answer to (b) is affirmative for the first case and negative for the second.

2. *Integrals of Functions not Assumed to be Factorable.*

THEOREM 1. *A necessary (as well as sufficient) condition that the restricted integral* (2) *shall exist for every continuous* $f(x, y)$ *is that* $\phi \subset V$.

PROOF. Assuming that $\phi(x, y)$ does not satisfy this condition, we shall show that there always exists a continuous function $f(x, y)$ such that (2) does not exist.

Clearly there must exist in R at least one point P having the following property: in at least one of the four quadrants about P, ϕ is not $\subset V$ in every rectangle with sides parallel to the axes and one vertex at P. Without loss of generality we may and do assume that this is the third quadrant. Hence we infer the existence of an infinite sequence of such rectangles $\{R_p\}$, each subdivided into cells by a net N_p of lines parallel to the axes, such that

(i) the lower left corner of R_{p+1} lies in the interior of the upper right cell of N_p;

(ii) $$\sum_{(N_p)} \left| \Delta_{11}\phi(x_i, y_j) \right| > p, \qquad (p=1, 2, 3, \cdots).$$

Now define the function $f(x, y)$ as follows. At the lower left vertex of each cell of N_p, $(p=1, 2, 3, \cdots)$, let $f(x, y) = +1/p$ or $-1/p$, according as $\Delta_{11}\phi(x_i, y_j)$ is $\geqq 0$ or <0 for that cell. Let each of these vertices be surrounded by a small square $S_{ij}^{(p)}$ in such a way that all the $S_{ij}^{(p)}$ lie below a horizontal line which in turn lies below all the $S_{ij}^{(p+1)}$, and to the left of a vertical line which in turn lies to the left of all the $S_{ij}^{(p+1)}$. Now let $f(x, y)$ vanish at any point which is not in the interior of some $S_{ij}^{(p)}$, and be so defined within these squares as to be continuous over R, which is clearly possible.

Then the integral (2) does not exist for this function. For assume that it does, and let A be its value. Then there exists a $\delta > 0$ such that for any net on R, the norm of whose subdivisions is $< \delta$, we have $|A - S| < 1/2$, where S is formed for this net, and any choice whatever of the intermediate points ξ_i, η_j. Let such a net N be fixed. Add to N the horizontal and vertical lines through P to form the net N', and to N' add the lines, suitably extended, of the net N_k, where k is chosen sufficiently large so that the squares $S_{ib}^{(k)}$ associated with the net N_k lie in the interior of that cell of N' which has P for upper right vertex. Let N'' be the resulting net.

Let S_1 be the sum S formed for the net N', where the points ξ_i, η_j are so chosen that $f(\xi_i, \eta_j) = 0$ in all terms of S associated with those cells of N' which lie in the column immediately to the left of P, and the row immediately below P; aside from this the choice may be arbitrary. Let S_2 be the sum S formed for the net N'', with the ξ_i, η_j chosen thus: in those columns of N'' which are also columns (extended) of N_k, let the ξ_i be selected as the left-hand point of each interval. In the next column to the left of these select ξ_i such that $f(\xi_i, y) = 0$ for all values of y. Analogously, in those rows of N'' which are also rows (extended) of N_k, let the η_j be selected at the bottom of each row; and in the row immediately below these, select η_j such that $f(x, \eta_j) = 0$ for all x. Let the remaining points be chosen as in S_1.

Then as the norms of both N' and N'' are $< \delta$, we have $|A - S_1| < 1/2$, $|A - S_2| < 1/2$, and hence $|S_1 - S_2| < 1$; but with the specified choice of nets and intermediate points, $S_2 - S_1 > 1$, with which contradiction the proof of Theorem 1 is complete.

COROLLARY 1. *A necessary (as well as sufficient) condition that the unrestricted integral* (3) *shall exist for every continuous* $f(x, y)$ *is that* $\phi \subset V$.

PROOF. The sufficiency of the condition may be shown by a method which does not differ in any essential from Fréchet's existence proof for the restricted integral.† The necessity follows from Theorem 1. From Theorem 1 and Corollary 1 we have at once the following result.

† See Fréchet, Nouvelles Annales, loc. cit.

COROLLARY 2. *If for a given $\phi(x, y)$ the restricted integral exists for every continuous integrand, then the unrestricted integral must exist for every continuous integrand.*

3. *Integrals of Factorable Functions.*

THEOREM 2. *A necessary (as well as sufficient) condition that the restricted integral*

$$\int_a^b \int_c^d g(x)h(y)d_x d_y \phi(x, y) \tag{4}$$

shall exist for every pair of continuous functions $g(x)$ and $h(y)$ is that $\phi \subset F$.

PROOF. Assuming that ϕ is not $\subset F$, we shall show that there always exists a pair of continuous functions $g(x)$, $h(y)$ such that (4) does not exist.

As before, we see that there must exist at least one point P (x_0, y_0) in R with the following property: in at least one of the four quadrants about P, ϕ is not $\subset F$ in every rectangle with sides parallel to the axes and one vertex at P. Again we assume the quadrant to be the third. Hence there exists an infinite sequence of such rectangles $\{R_p\}$, on each of which there exists a net N_p, such that

(i) the lower left vertex of R_{p+1} lies in the interior of the upper right cell of N_p;

(ii) $$\sum_{N_p} \epsilon_i \bar{\epsilon}_j \, \Delta_{11}\phi(x_i, y_j) > p^3, \qquad (p = 1, 2, 3, \cdots),$$

for some choice of the ϵ_i's and $\bar{\epsilon}_j$'s as $+1$ or -1.

Let m_p, n_p be the number of rows and columns, respectively, in the net N_p. Let $I_i^{(p)}$, $(i = 1, 2, 3, \cdots, n_p - 1)$, be the segments cut off on the x-axis by the $(n_p - 1)$ left-hand-most columns, extended, of N_p, numbering serially from the left; and let $I_{n_p}^{(p)}$ be the interval lying between $I_{n_p-1}^{(p)}$ and $I_1^{(p+1)}$.

We may now define $g(x)$ as follows. Interior to the interval $I_i^{(p)}$, $(i = 1, 2, 3, \cdots, n_p; p = 1, 2, 3, \cdots)$, select any two distinct points x_{ip} and x_{ip}'; let $g(x)$ at these points assume the values $+1/p$ and $-1/p$, respectively. Let $g(x)$ vanish for $x \geqq x_0$, and be so defined for remaining values of x as to be continuous in the interval $a \leqq x \leqq b$.

Let $J_j^{(p)}$, $(j = 1, 2, 3, \cdots, m_p; p = 1, 2, 3, \cdots)$, denote the analogous set of segments on the y-axis, and define the function $h(y)$ in precisely similar fashion, the points analogous to x_{ip} and x_{ip}' being denoted by y_{jp} and y_{jp}'.

Now suppose that the integral (4) exists with this choice of g and h. Let $\epsilon(>0)$ be fixed. Then there exists a $\delta>0$ such that for any net N on R the norm of whose sub-divisions is $<\delta$, and any choice whatever of the intermediate points ξ_i, η_j, the inequality

$$| S - A | = | \sum_{(N)} g(\xi_i)h(\eta_j)\Delta_{11}\phi(x_i, y_j) - A | < \epsilon$$

is satisfied, where A is the value of (4). Let N, such a net, be fixed. Add to N the horizontal and vertical lines through P to form the net N'. Let k be any integer $>2\epsilon$ and such that the lower left vertex of R_0 lies within the cell of N' whose upper right vertex is P. Add to N' the lines of N_k, suitably extended, to form the net N''.

Let the sum S be formed for the net N''. In those rows of cells of N'' which are not also rows (extended) of N_k let the points η_j be chosen arbitrarily; and also let an arbitrary choice be made of those ξ_i which attach to all columns of N'' except those which are also columns (extended) of N_k. In view of condition (ii) we may clearly select the remaining ξ_i and η_j from among the points x_{ik}, x_{ik}', y_{jk}, y_{jk}' in such a manner that B, the part contributed to S by the cells of N_k, shall exceed k. Let S_1 denote the resulting value of the sum S. Let C denote that part of S_1 which is contributed by cells of N'' lying in rows of N'' that are also rows (extended) of N_k, excepting the cells of N_k; let D denote the part contributed by cells of N'' which occur in columns of N'' that are at the same time columns (extended) of N_k, excepting the cells of N_k; let E denote the remaining part of S_1. Then we have $S_1=B+C+D+E$ and

$$|A - B - C - D - E| < \epsilon.$$

Now form the sum S_2 by using the same net N'' and the same choice of the points ξ_i, η_j, except that those of the ξ_i which were selected from the x_{ik} and x_{ik}' are to be altered as follows: in every case where formerly a point x_{ik} was chosen, to form S_2 choose instead the point x_{ik}' which occurs in the same interval $I_i^{(k)}$, and vice versa. Then we clearly have $S_2=-B+C-D+E$, and $|A+B-C+D-E|<\epsilon$.

By repeated use of this device we easily see that the following inequalities are also satisfied:

$$|A + B + C - D - E| < \epsilon,$$
$$|A - B + C + D - E| < \epsilon.$$

From these inequalities we have at once

$$(5) \qquad |B + D| < \epsilon,$$

$$(6) \qquad |B + C| < \epsilon,$$

$$(7) \qquad |C + D| < \epsilon.$$

But $B > k > 2\epsilon$; hence from (5) and (6) we have $D < -\epsilon$, $C < -\epsilon$, whence $|C+D| > 2\epsilon$; and as this contradicts (7), the theorem is proved.

THEOREM 3. *A necessary (as well as sufficient) condition that the unrestricted integral*

$$(8) \qquad (*)\int_a^b \int_c^d g(x)h(y)d_x d_y \phi(x, y)$$

shall exist for every pair of continuous functions $g(x)$, $h(y)$ *is that* $\phi \subset V$.

PROOF. Again we assume ϕ is not $\subset V$, and by the same argument as before show the existence of an infinite sequence of rectangles $\{R_p\}$ with nets N_p as in the proof of Theorem 2, the condition (ii) given there being now replaced by the condition

$$\text{(iii)} \qquad \sum_{(N_p)} |\Delta_{11}\phi(x_i, y_j)| > p^3, \qquad (p = 1, 2, 3, \cdots).$$

Then the functions $g(x)$ and $h(y)$ defined in that theorem will serve as a pair of functions for which the unrestricted integral (8) will not exist. Since for each p there is in every cell of the net N_p a point at which $g \cdot h = 1/p^2$, and a point at which $g \cdot h = -1/p^2$, the argument is sufficiently clear without giving the details of the proof.

COROLLARY. *If for a given* $\phi(x, y)$ *the unrestricted integral* (3) *exists for every factorable integrand with continuous factors, it must exist for every continuous integrand.*

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