ON THE INFINITE SEQUENCES ARISING IN THE THEORIES OF HARMONIC ANALYSIS, OF INTERPOLATION, AND OF MECHANICAL QUADRATURES*

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1. *Introduction.* The three mathematical theories indicated in my title are so extensive that I should naturally be unwilling within the bounds of a single discussion to give an outline of the totality of the relevant investigations. On the contrary I shall in each case bind myself to a portion of the corresponding theory. The investigations which I have in mind, and which I hope to be able to present to you, have been conducted almost entirely in the twentieth century. Even in this portion of the theory, however, so many brilliant contemporary mathematicians have collaborated that a considerable complex of investigations has resulted. Thus I shall select from this narrower field only a few results,—such, however, as are characteristic and have served as points of departure for further researches.

I shall therefore undertake to give only an outline of these dominant characteristic results, and shall accomplish this by exhibiting as clearly as possible the *single fundamental idea* which unites them. If I can succeed in the course of my lecture in making the investigations of the whole complex seem to you less diversified, I shall have achieved my goal.

2. *Fourier Series.* I begin my exposition with Fourier series. If $f(t)$ denotes an integrable real function of the real variable t, having the period 2π, then the constants

$$(1)\qquad \begin{cases} a_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(t)dt, & \\ a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\cos nt\, dt, & \\ & (n = 1, 2, 3, \cdots), \\ b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\sin nt\, dt, & \end{cases}$$

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are known as the Fourier constants of $f(t)$, and the infinite series

$$a_0 + a_1 \cos x + b_1 \sin x + \cdots \tag{2}$$
$$+ a_n \cos nx + b_n \sin nx + \cdots = \sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called the Fourier series of the function f. The sum of the first $n+1$ terms of this series is clearly

$$s_n(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\{1 + 2 \cos t \cos x + 2 \sin t \sin x + \cdots \tag{3}$$
$$+ 2 \cos nt \cos nx + 2 \sin nt \sin nx\} dt.$$

In most cases, however, it is convenient to note that

$$1 + 2 \cos t \cos x + 2 \sin t \sin x + \cdots$$
$$+ 2 \cos nt \cos nx + 2 \sin nt \sin nx = \frac{\sin (2n+1)\dfrac{t-x}{2}}{\sin \dfrac{t-x}{2}},$$

so that we have for the so-called partial sum of the Fourier series of index n for the value x the Dirichlet formula

$$s_n(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\frac{\sin (2n+1)\dfrac{t-x}{2}}{\sin \dfrac{t-x}{2}} dt. \tag{4}$$

Now let us form the arithmetic mean $S_n(x)$ of the partial sums $s_0(x), s_1(x), \cdots, s_n(x)$, so that

$$S_n(x) = \frac{s_0(x) + s_1(x) + \cdots + s_n(x)}{n+1}. \tag{5}$$

From the obvious formula

$$S_n(x) = \frac{1}{n+1}\,\frac{1}{2\pi}\int_0^{2\pi} f(t)\{(n+1) + n\cdot 2 \cos t \cos x \tag{6}$$
$$+ n\cdot 2 \sin t \sin x + \cdots + 1\cdot 2 \cos nt \cos nx$$
$$+ 1\cdot 2 \sin nt \sin nx\} dt,$$

by use of the identity

$$(7)\quad (n+1) + n\cdot 2\cos t\cos x + n\cdot 2\sin t\sin x + \cdots$$
$$+ 1\cdot 2\cos nt\cos nx + 1\cdot 2\sin nt\sin nx = \left\{\frac{\sin (n+1)\dfrac{t-x}{2}}{\sin\dfrac{t-x}{2}}\right\}^2,$$

follows

$$(8)\qquad S_n(x) = \frac{1}{2\pi(n+1)}\int_0^{2\pi} f(t)\left\{\frac{\sin (n+1)\dfrac{t-x}{2}}{\sin\dfrac{t-x}{2}}\right\}^2 dt,$$
$$(n = 0, 1, 2, \cdots).$$

If to each value x of an interval is made to correspond a numerical value of f, f is called a function of x. If to each function f of a set of functions a numerical value A is made to correspond, then A is said to be obtained from f by a functional operation (briefly *operation*). If we now think of x as fixed, and correspondingly write s_n, S_n for $s_n(x)$, $S_n(x)$, then obviously s_n and S_n are each formed by an operation on $f(t)$:

$$(9)\qquad s_n = s_n[f],$$

$$(10)\qquad S_n = S_n[f].$$

These operations are both ordinary linear functional operations. An essential difference between them is, however, that the operation $S_n[f]$ is *positive*, while the classical operation $s_n[f]$ does *not* possess this property. *An operation* $A[f]$ *is called positive, provided* $A[f] \geqq 0$ *whenever* $f(t) \geqq 0$ *throughout the interval of values of* t *in which the function* $f(t)$ *is considered.*

I published this result in 1900. It is remarkable that from the classical *indefinite* operation $s_n[f]$, the *definite* (positive) operation $S_n[f]$ can be obtained by a very slight modification without any limit process. The sharpness of the contrast between these two types of linear operation, as viewed at the time of my investigation, undoubtedly increased the long existing interest of mathematicians in this distinction.

So much has been published concerning the meaning of the operation $S_n[f]$ of the arithmetic mean, and various kinds of more delicate considerations, further generalizations, and applications, that I must unfortunately omit all mention of them, in accordance with the plan of this lecture. I must thus neglect a mass of material, to which I have myself often endeavored to contribute, and which has been enriched by a long list of profound articles by other mathematicians.

3. *Further Remarks on Fourier Series.* I continue, still remaining within the domain of Fourier series. As we have essentially seen already, the positiveness of the operation $S_n[f]$ follows immediately from the simple fact that the ordinary arithmetic means of the partial sums of the series

$$1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos n\theta + \cdots \tag{11}$$

are all non-negative for every real value of θ. Very recently I have made the observation that for the series

$$0 + \sin\theta + 2\sin 2\theta + 3\sin 3\theta + \cdots + n\sin n\theta + \cdots \tag{12}$$

the arithmetic means of third order $S_n^{(3)}(\theta)$ of the partial sums $S_n^{(0)}(\theta) = s_n(\theta)$, are all positive in the interval $0<\theta<\pi$ (except of course $S_0^{(3)}(\theta)\equiv 0$). For the means of orders 0, 1, and 2 of the series (12) this assertion is *not* valid. I have further noted,—and this seems to lie somewhat deeper,—that for the series

$$\sin\theta + 3\sin 3\theta + 5\sin 5\theta + \cdots \tag{13}$$

even the arithmetic means of second order of the partial sums are all non-negative in the interval $0<\theta<\pi$. (In this case even more detailed statements can be made.) For the means of orders 0 and 1 of the series (13) this assertion is *not* valid.

To these properties of the series (12) and (13) again correspond in a certain sense *positive linear functional operations* on the pure Fourier sine-series or the pure cosine-series of a function $f(x)$. But I shall not exhibit these operations here, nor discuss the numerous interesting consequences of their positiveness.

As, however, this is an entirely new direction of investigation, I shall allow myself to mention just one characteristic result.

Let the function $f(x)$ be positive and convex (or even not concave) upward everywhere in the interval $0<x<\pi$. In order to have a particularly interesting special case, I assume further that the curve $y=f(x)$ has the property of symmetry,

$$f(\pi - x) = f(x) \quad \text{for} \quad 0 \leqq x < \frac{\pi}{2}.$$

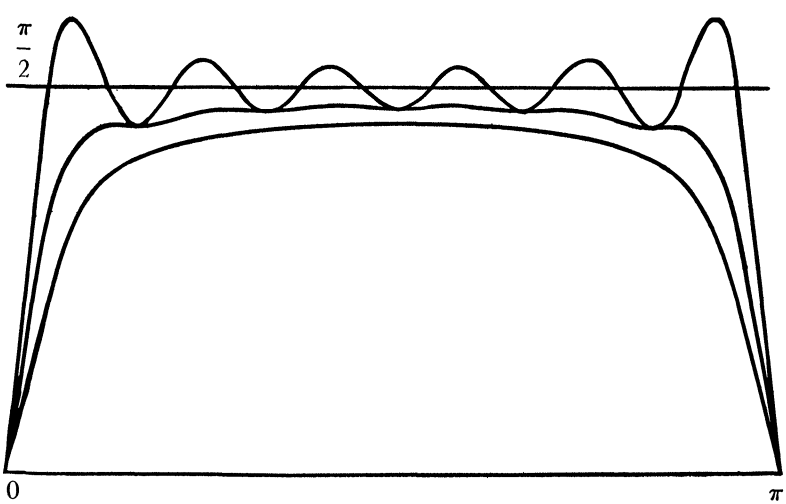
The Fourier sine-series of the function $f(x)$ has the form

$$(14) \quad f(x) \sim b_1 \sin x + b_3 \sin 3x + \cdots + b_{2\nu-1} \sin (2\nu - 1)x + \cdots,$$

where

$$(15) \qquad b_{2\nu-1} = \frac{2}{\pi}\int_0^\pi f(t) \sin (2\nu - 1)t dt, \quad (\nu = 1, 2, 3, \cdots).$$

As an immediate consequence of the statement about the series (13) we have the following theorem: all arithmetic means of second order of the partial sums of the Fourier sine-series (14) for the function $f(x)$ are positive and convex upward in the interval $0<x<\pi$. The arithmetic means of orders 0 and 1 on the other hand do *not* in general possess this property of convexity.



The accompanying figure* illustrates this theorem in the simplest imaginable special case, in which $f(x)$ is constant in the

* I am indebted to Mr. I. Raisz for the careful drawing of this figure.

interval $0 \leqq x \leqq \pi$ (thus $y=f(x)$ is non-concave upward). The Fourier sine-series of the function

$$f(x) = \frac{\pi}{2}, \qquad (0 < x < \pi), \tag{16}$$

is

$$\begin{aligned} f(x) = \frac{\pi}{2} = 0 + \frac{\sin x}{1} + 0\frac{\sin 2x}{2} + \frac{\sin 3x}{3} + 0\frac{\sin 4x}{4} \\ + \frac{\sin 5x}{5} + \cdots . \end{aligned} \tag{17}$$

The upper curve of this figure represents the partial sum of index 12 of this sine-series (17), that is,

$$S_{12}^{(0)}(x) = \sum_{\nu=1}^{6} \frac{\sin (2\nu - 1)x}{2\nu - 1} .$$

The middle curve represents the arithmetic mean of first order $S_{12}^{(1)}(x)$ and the lower curve the Cesàro mean of second order $S_{12}^{(2)}(x)$ of the series (17). The interesting curves $S_{12}^{(0)}(x)$, $S_{12}^{(1)}(x)$ are already found in the book of H. S. Carslaw on Fourier Series in connection with the discussion of the famous Gibbs phenomenon, which arises in the case of the partial sums $s_n(x) = S_n^{(0)}(x)$ but disappears for the means of first order $S_n^{(1)}(x)$. If we now examine the three curves from the point of view of convexity-concavity, we observe that both $S_{12}^{(0)}(x)$ and $S_{12}^{(1)}(x)$ are composed of arcs, which are alternately convex and concave upward. On the other hand the new curve $S_{12}^{(2)}(x)$, the lower curve, which represents the Cesàro mean of second order of index 12 for the series (17), is convex upward in the whole interval $0 < x < \pi$.

Theorems such as the preceding illustrate from a new angle the "smoothing-out effect" on the Fourier series of the process of forming repeated means. I should like to conclude this consideration with only the further observation that the smoothing-out effect of repeated formation of means is very notable in another connection. If

$$w = f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots \tag{18}$$

is a power series which is regular and assumes each of its values only once in the circle $|z|<1$, this property is not in general shared by the partial sums

$$(19)\qquad S_n^{(0)}(z) = s_n(z) = c_0 + c_1 z + \cdots + c_n z^n, \quad (n=0, 1, 2, 3, \cdots),$$

for the whole unit circle. But the arithmetic means of second or third order $S_n^{(2)}(z)$, $S_n^{(3)}(z)$ of the partial sums (19) maintain the property of the original function (i.e. take on each of their values only once in the whole unit-circle), provided the power series $c_0+c_1z+c_2z^2+\cdots$ belongs to a certain special, but remarkable, subclass of the totality of series having the specified property.

4. *Laplace Series.* If $S_n^{(0)}[f]=s_n[f]$ denotes the partial sum of index n of the Laplace series for a function $f(\theta, \phi)$, integrable over the unit sphere, then it is well known that for a fixed point (θ, ϕ) of the unit sphere $S_n^{(0)}[f]$ represents an indefinite linear operation on f. The operation $S_n^{(1)}[f]$, which is defined by the arithmetic means of first order of the Laplace series, is also indefinite. But the operation $S_n^{(2)}[f]$ is *positive.*

I obtained this result in 1908. It is now possible to give a proof as follows. Obviously, we have only to show that the means of second order for the series

$$(20)\qquad P_0(\cos\theta) + 3P_1(\cos\theta) + 5P_2(\cos\theta) + \cdots + (2n+1)P_n(\cos\theta) + \cdots$$

are positive for $0<\theta<\pi$. This is accomplished at one stroke by the use of Mehler's formula for the Legendre polynomial $P_n(\cos\theta)$:

$$(21)\qquad P_n(\cos\theta) = \frac{2}{\pi}\int_\theta^\pi \frac{\sin\,(2n+1)\dfrac{t}{2}}{[2(\cos\theta-\cos t)]^{1/2}}dt,$$

since, as previously stated, the means of second order of the series

$$(22)\qquad \sin\frac{t}{2} + 3\sin 3\frac{t}{2} + 5\sin 5\frac{t}{2} + \cdots + (2n+1)\sin\,(2n+1)\frac{t}{2} + \cdots$$

are positive for $0<t<\pi$.

I shall not dwell further on positive operations in the domain of Fourier and Laplace series. But in any case I must at least refer to the two important positive operations introduced by D. Jackson. Both are closely related to our first example of a linear operation, that of the arithmetic mean, in which occurs the "kernel" $(\sin nt/\sin t)^2$. Jackson's first operation is used for the approximation (by trigonometric polynomials) to functions subjected to certain restrictions. His second operation is used for the trigonometric approximation to arbitrary continuous functions; it consists in a method of interpolation, and is closely related to the operations which we shall consider next.

5. *Interpolation.* In the present part of this lecture I shall concern myself with interpolation, and in particular, I shall indicate briefly how the theory of interpolation can utilize the idea of the positive linear functional operation.

If $x_1, x_2, \cdots, x_n$ denote arbitrary distinct real numbers, then the Lagrange interpolation formula is

$$L(x) = y_1 l_1(x) + y_2 l_2(x) + \cdots + y_n l_n(x). \tag{23}$$

This formula represents the polynomial of degree at most $n-1$, which assumes the arbitrarily given values $y_1, y_2, \cdots, y_n$ respectively for the values $x_1, x_2, \cdots, x_n$. Here

$$l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \quad (k = 1, 2, \cdots, n), \tag{24}$$

where

$$\omega(x) = C(x - x_1)(x - x_2) \cdots (x - x_n), \qquad (C \neq 0). \tag{25}$$

Usually $y_1, y_2, \cdots, y_n$ are the ordinates of a given curve $y=f(x)$ for the abscissas $x=x_1, x_2, \cdots, x_n$; that is, $y_1=f(x_1)$, $y_2=f(x_2), \cdots, y_n=f(x_n)$. In this sense, therefore, the classical Lagrange formula defines for a fixed value of x a linear functional operation $L[f]$. For *no* choice of the abscissas $x_1, x_2, \cdots, x_n$ (if $n \geqq 2$), however, is this operation *positive* (for any fixed value of x different from $x_1, x_2, \cdots, x_n$). This is because each of the so-called fundamental polynomials $l_1(x), l_2(x), \cdots, l_n(x)$ of the Lagrange interpolation formula changes its sign $n-1$ times.

Let us now examine from this point of view the simplest so-called Hermite interpolation formula

$$(26) \qquad X(x) = \sum_{k=1}^{n} y_k h_k(x) + \sum_{k=1}^{n} y_k' \mathfrak{h}_k(x).$$

This represents the polynomial of degree at most $2n-1$ which for the values $x_1, x_2, \cdots, x_n$ assumes respectively the values $y_1, y_2, \cdots, y_n$ and whose derivative correspondingly assumes the values $y_1', y_2', \cdots, y_n'$. Here $h_1(x), h_2(x), \cdots, h_n(x)$ are the so-called "fundamental functions of the first kind" for the Hermite interpolation formula (26):

$$(27) \quad h_k(x) = \left(1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k)\right)(l_k(x))^2, \quad (k = 1, 2, \cdots, n),$$

and $\mathfrak{h}_1(x), \mathfrak{h}_2(x), \cdots, \mathfrak{h}_n(x)$ are the "fundamental functions of the second kind":

$$(28) \qquad \mathfrak{h}_k(x) = (x - x_k)(l_k(x))^2, \quad (k = 1, 2, \cdots, n),$$

where $\omega(x)$ and $l_k(x)$ have the same meaning as before.

If we now merely glance at the Hermitian fundamental functions $h_k(x)$ and $\mathfrak{h}_k(x)$, we see at once that they have, in contrast to the Lagrangian fundamental functions $l_k(x)$, a kind of tendency to be positive. I shall make this statement more precise.

The fundamental function $\mathfrak{h}_k(x)$ is equal to the square of the polynomial of $(n-1)$th degree $l_k(x)$ multiplied by the linear function

$$(29) \qquad w_k(x) = x - x_k.$$

Hence $\mathfrak{h}_k(x)$ changes its sign exactly once, at the interpolation point x_k.

The fundamental function $h_k(x)$ is equal to the square of $l_k(x)$ multiplied by the linear function

$$(30) \qquad v_k(x) = 1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k).$$

Thus $h_k(x)$ also can change its sign only once, at the vanishing point X_k of the linear function $v_k(x)$,

$$(31) \qquad X_k = x_k + \frac{\omega'(x_k)}{\omega''(x_k)} \cdot$$

The point X_k is however, under all circumstances, different from x_k, since $v_k(x_k)=1$. If the interpolation points

$$x_1,\ x_2,\ \cdots,\ x_n \tag{32}$$

are given, the points

$$X_1,\ X_2,\ \cdots,\ X_n \tag{33}$$

are uniquely determined. I shall call X_k the *conjugate point* corresponding to the interpolation point x_k.

Henceforward let the interpolation points $x_1, x_2, \cdots, x_n$ lie in the interval $-1 \leqq x \leqq 1$, and let x range over this interval only. We then obtain the following result:

The Hermite fundamental polynomials of the first kind are non-negative in the whole interval of interpolation $-1 \leqq x \leqq 1$, *if and only if all the conjugate points* $X_1, X_2, \cdots, X_n$ *lie outside the interval of interpolation* $-1 < x < 1$.

If we now consider the linear functional operation

$$H(x) = \sum_{k=1}^{n} f(x_k) h_k(x), \tag{34}$$

whose meaning is easily discerned, we can make the following restatement:

A necessary and sufficient condition for the positiveness of the linear operation (34) *for every fixed value of x in the interval* $-1 \leqq x \leqq 1$ *is that the interval* $-1 < x < 1$ *be free from conjugate points.*

By the assumption of the positiveness of a linear functional operation (the operation (34)) we are thus led to the classification of all point-systems $x_1, x_2, \cdots, x_n$ of the interval $-1 \leqq x \leqq 1$ into two classes. To the first class belong those systems $x_1, x_2, \cdots, x_n$ for which the conjugate points $X_1, X_2, \cdots, X_n$ all lie outside the interval $-1 < x < 1$; to the second class belong all other systems.

The importance of this distinction which rests on our postulate of positiveness of the operation lies primarily in the fact that those point-systems $x_1, x_2, \cdots, x_n$ which are useful for interpolation and are of greatest mathematical interest belong to our first class.

For instance let $x_1, x_2, \cdots, x_n$ be the Tschebyscheff abscissas, which present themselves from so many modes of approach. They may be obtained by describing a semicircle having as diameter the interval from -1 to $+1$ on the x-axis, dividing this semicircle into n equal arcs, and projecting the points of bisection of these arcs on the x-axis.

In this special case it is easily seen that the point X_k conjugate to x_k is $X_k = 1/x_k$; i.e., X_k is the harmonic conjugate to the interpolation point x_k with respect to the points -1, $+1$. Since the harmonic conjugates to $x_1, x_2, \cdots, x_n$ all lie outside the interval $(-1, +1)$, it is clear that the Tschebyscheff abscissas $x_1, x_2, \cdots, x_n$ belong to our first class of point-systems.

As a second example we choose the point-system which was introduced by Gauss in his famous monograph on mechanical quadrature by parabolic interpolation. The points of interpolation $x_1, x_2, \cdots, x_n$ are now the roots of the equation

$$P_n(x) = 0, \tag{35}$$

where $P_n(x)$ denotes the Legendre polynomial of index n. It is readily seen that in this case $X_k = (x_k + 1/x_k)/2$; that is, the conjugate point X_k is half-way between the point of interpolation x_k and its harmonic conjugate. But the point of bisection of the interval between two conjugate harmonic points always lies outside the interval between the fundamental points. The set of Legendre-Gauss abscissas thus belongs to our first class of point-systems.

More generally the roots $x_1, x_2, \cdots, x_n$ of the equation

$$J_n(\alpha, \beta, x) = 0 \tag{36}$$

always belong to the first class, provided that $J_n(\alpha, \beta, x)$ denotes the so-called nth Jacobi polynomial and the parameters α, β satisfy the inequalities $0 \leqq \alpha \leqq \frac{1}{2}$, $0 \leqq \beta \leqq \frac{1}{2}$. For $\alpha = \beta = \frac{1}{2}$ this set of roots is that of Gauss, for $\alpha = \beta = \frac{1}{4}$ that of Tschebyscheff. I call attention to a third special case of exceptional interest: $\alpha = \beta = 0$. In this case the points of interpolation are -1, $+1$, and the $n-2$ roots of the equation $P_{n-1}'(x) = 0$ where $P_{n-1}(x)$ denotes the Legendre polynomial of index $n-1$.

For infinite sequences of Lagrange and Hermite interpolation polynomials of a function, provided the point-systems always belong to the first class, it is now possible to state very general theorems, all of which can be proved with surprising ease. But I cannot dwell further on this important subject. I proceed to my third and last topic—mechanical quadrature by parabolic interpolation.

6. *Mechanical Quadrature.* Let $f(x)$ be a function defined and Riemann-integrable in the interval $-1 \leqq x \leqq 1$. If $x_1, x_2, \cdots, x_n$ again denote n distinct points of the interval $-1 \leqq x \leqq 1$, then

$$L(x) = \sum_{k=1}^{n} f(x_k) l_k(x) \tag{37}$$

is the corresponding Lagrange parabola and the value of the integral

$$Q = \int_{-1}^{+1} L(x)dx = \sum_{k=1}^{n} f(x_k) \int_{-1}^{+1} l_k(x)dx \tag{38}$$

is called the corresponding *mechanical quadrature*. The factors

$$\lambda_1 = \int_{-1}^{+1} l_1(x)dx,\ \lambda_2 = \int_{-1}^{+1} l_2(x)dx, \cdots, \lambda_n = \int_{-1}^{+1} l_n(x)dx, \tag{39}$$

the integrals of the Lagrange fundamental functions over the interval of quadrature, which depend only on the abscissas $x_1, x_2, \cdots, x_n$, I call the Cotes numbers. It is clear that the quadrature formula

$$Q = Q[f] = \sum_{k=1}^{n} \lambda_k f(x_k) \tag{40}$$

again represents a linear functional operation; and I remark at once that it is in general indefinite, i.e., there are both positive and negative numbers among the Cotes numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$. This is true, for instance, if for an infinite set of values of n the points $x_1, x_2, \cdots, x_n$ divide the interval $(-1, +1)$ into n equal parts. In this connection I may cite the important investigations of Uspensky and Pólya, and I should like to remark that the results of the latter have given a new impulse to this direction of study.

From what has preceded one is led almost involuntarily to a new classification of point-systems $x_1, x_2, \cdots, x_n$ of the interval $(-1, +1)$. In the first class are placed those systems $x_1, x_2, \cdots, x_n$ for which the Cotes numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ are all non-negative, in the second class all other systems.

Does there exist even one single point-system $x_1, x_2, \cdots, x_n$ which belongs to the first class? This question is answered in the affirmative by the classical result of Gauss, Christoffel, and Stieltjes; for the Legendre abscissas $x_1, x_2, \cdots, x_n$ the Cotes numbers are all positive, i.e., the operation of quadrature is positive. I have recently discovered that the Cotes numbers are positive also for the Tschebyscheff abscissas; for the Jacobi abscissas if $\alpha=\beta=0$ or $\alpha=\beta=\frac{3}{4}$ or $\alpha=\frac{1}{2}, \beta=0$, and for other systems as well. Subsequently Szegö in an original fashion has treated the question of positiveness of the Cotes numbers for the most general Jacobi systems.

Numerous theorems of very general character can now be proved with the utmost ease, dealing with point-systems for which the operation of quadrature is definite, and with the corresponding quadratures themselves. I shall mention one such theorem: if we have an infinite sequence of such point-systems, consisting of more and more points, then the corresponding quadratures converge to $\int_{-1}^{+1} f(x)dx$ as $n=\infty$, if $f(x)$ is bounded and Riemann-integrable in the interval. This theorem is a far-reaching generalization of the theorem of Stieltjes, which enunciates the same result on convergence for the special case of the Legendre-Gauss point-system. But I consider it remarkable that it has apparently been overlooked hitherto that such special systems, as for instance that of Tschebyscheff, also belong to our first category of point systems.

We have divided point systems $x_1, x_2, \cdots, x_n$ of the interval $-1 \leqq x \leqq 1$ sharply into two classes in connection with interpolation; we have again divided these systems sharply into two classes in a different way in connection with mechanical quadrature. May these two classifications be characterised in any other way? What relation subsists between the two classifications? I content myself with a mere mention of these problems.

Just one word more about the case of quadrature. For

Fourier and Laplace series and for interpolation the original classical expressions (for instance the partial sums of the series, the Lagrange polynomials of the interpolation) are indefinite; only among the modified expressions (arithmetic means, Hermite interpolation polynomials, etc.) do we find those which are positive. For quadrature, on the other hand, we have seen that the original quadrature, obtained by means of the classical Lagrange parabola, is *positive*, provided the abscissas of the quadrature belong to a certain well defined class.

7. *Conclusion.* I have come to the end of my lecture, to which I might as well have given the following title: On the significance of the idea of the positive linear operation for harmonic analysis, interpolation, and quadrature. I should like to add just one brief remark, relating to the whole subject-matter of the lecture. In each case we were concerned with a set of linear operations, from which we selected those that were positive, in order to arrive at certain goals appropriate to the various theories. Is it actually necessary, in order to arrive at these goals, to demand that the operations be positive? By no means. In order to obtain necessary and sufficient conditions, a consideration of absolute values is demanded, as in the familiar work of Lebesgue in breaking the path for the study of singular integrals. Then why have we concerned ourselves so particularly with positive operations? Because the most elementary operations which we meet at the outset in the three fields are fortunately positive. And for another reason as well: if we do not seek complete generality, but subject the operations to certain natural supplementary restrictions, then positiveness of the operations is both necessary and sufficient.

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