as belonging to one of a finite number of non-equivalent types. This meager knowledge depends partly on the empirical results of Tait and his contemporaries, partly on the known calculable knot invariants such as those discovered by Alexander, and partly on special results (some of which are still unpublished) concerning pairs of knots which are not distinguishable by their invariants. Thus the problem is still open and still fascinating—the more so since it is now apparent that even though the problem seems to be one of abstract groups, progress may depend on the results of the most unexpected domains of algebra. The complete and concise little work of Reidemeister will do much to encourage further attacks.

P. A. SMITH

MOORE ON POINT SETS

Foundations of Point Set Theory. By R. L. Moore. American Mathematical Society Colloquium Publications, Volume 13. New York, 1932. viii +486 pp.

We are told in the Preface that this volume is intended to be a self-contained treatment of the foundations of the point-set-theoretic branch of analysis situs. It is concerned chiefly with those topics which are the results of Professor Moore's own research. Hence the book does not mention certain topics—dimension theory, for instance—which are closely connected with the topics discussed, but with the development of which Professor Moore was not primarily concerned.

The present treatment of point set theory is based upon a system of axioms, the undefined notions being *point* and *region*. General logical concepts are assumed, including in particular those of the logic of classes and the fundamental propositions concerning integers. Of these concepts, the author states explicitly only the Zermelo Axiom, which is to be considered as included among his assumptions. In addition to these introductory statements, the Introduction (pp. 1–4) is concerned with definitions of various types of sequences.

Chapter 1 (pp. 5–85) contains the theorems which can be derived from Axioms 0 and 1. The author also shows that most of these theorems follow from Axioms 0 and 1_0, where 1_0 is a weaker form of Axiom 1. Since Axioms 1 and 1_0 each consist of several parts, it is not surprising that so many theorems can be proved from them. While certain theorems can be proved with still weaker hypotheses, the author makes no attempt to do so. But the group of examples on pages 24–28 shows that in the case of certain theorems at least, no unnecessary restrictions have been placed in the hypotheses to compensate for the weakness of the assumptions concerning the space containing the sets. It is shown by these examples that under weaker hypothese the conclusions of this group of theorems become false, even if we assume that the underlying space is the euclidean plane.

The topics discussed in Chapter 1 are concerned with the following ideas, arranged roughly in order: boundary point, sequential limit point, Bo el property, connectivity, irreducible continuum, limiting set, separation of two sets by a third set, cut point, arc, link, indecomposable continuum, inner limiting set.

In Chapter 2 (pp. 86–151) are contained the consequences of Axioms 1 and 2. Some of the concepts defined and discussed in this chapter are: connectivity im kleinen, local connectivity, continuous curve, simple continuous curve, end point, acyclic continuous curve, regular curve, cyclic connectivity. From the standpoint of the study of continuous curves, Theorem 9 of this chapter is of importance, inasmuch as this theorem states that all the internal properties of a continuous curve may be derived from the fact that a continuous curve is a space satisfying Axioms 0, 1, 2.

Chapter 3 (pp. 152–174) discusses the consequences of Axioms 1–4. This chapter is principally concerned with simple closed curves and the sets complementary to them. The convention introduced on page 153 of selecting a definite point and calling it the point at infinity is a useful method to show that theorems concerning subsets of a sphere can be proved from corresponding theorems concerning subsets of a plane. If this convention is followed when the space S is a plane, the usual meaning of *exterior* and *interior* will be interchanged in the case of certain simple closed curves, and *bounded* and *unbounded* will not have their usual meaning. Hence proper care must be exercised in interpreting any theorems proved for the plane on the basis of this convention.

The theorems of Chapter 4 (pp. 175–323) are derived from Axioms 1–5. They may be divided roughly as follows: Theorems showing the existence of arcs bearing certain relations to given sets, theorems on separation by simple closed curves and by other continua, theorems on abutting and crossing of arcs, theorems on continuous curves and their complementary domains, theorems on accessibility, theorems on triods and related concepts, theorems on properties S and S' and their relation to uniform connectivity im kleinen. The chapter concludes with the proof of the Moore-Kline theorem on the conditions under which a closed set is a subset of an arc.

Chapter 5 (pp. 324–411) is entitled: *Upper semi-continuous collections.* The first part of this chapter is based upon Axioms 0, 1′, and C, where 1′ is a stronger form of Axiom 1. The second part is based on Axioms 1′, C, 2–6. Part 3 discusses equicontinuous collections of continua. Other concepts defined in this chapter are: graphatomic subset, order of a point, essential continuum of condensation.

In Chapter 6 (pp. 412–429) and Chapter 7 (pp. 430–461), the author shows that every space which satisfies a certain collection of the axioms is topologically equivalent to the number plane, and the substitution of Axiom 8′ for Axiom 8 gives a similar result for the number sphere. Theorems are also given on extending a homeomorphism between two sets to a homeomorphism between two sets containing the given sets. Finally the matter of introducing *distance* is taken up.

In an appendix (pp. 462–465) the author comments on certain theorems and their relation to the work of other mathematicians. The book concludes with an extensive bibliography and a glossary.

As in any book on the foundations of a branch of mathematics, the object of the author is to prove as much as possible from a minimum set of assumptions and to make his proofs as precise and as rigorous as possible. We shall now comment on a few minor inaccuracies which may serve to confuse the reader of the book.

The definition of a *sequence* on page 2, is phrased in a rather original manner. A sequence G is a collection of *sensed pairs*, whose elements are selected from a set M. (It might be noted parenthetically that a sensed pair, or ordered pair, is one of the most fundamental concepts underlying the use of logical symbolism.) The sensed pairs are the *elements* of the sequence G; the *terms* of the sequence are the elements of the set M. A term x is said to precede a term y in the sequence (although neither x nor y is an element of the sequence) if the sensed pair (x, y) is a member of the sequence. But in the final paragraph of page 4, when the author refers to the sequence $(A, 1)$, etc., the symbols are *not* intended to represent sensed pairs, as one might infer from the definition of sequence. The ambiguity inherent in this paragraph might have been avoided if the precise meaning of the symbols (A, i) and (B, j) had been given.

Sequence as thus defined means any type of arrangement in linear order. Various types of sequence are then defined: well-ordered sequence, infinite sequence, finite sequence, simple sequence. The reader is then told that *sequence* is to be used hereafter in the sense of simple sequence. It is unfortunate that some other word was not used for the general concept of sequence as first defined. It is rather confusing to use the same word for a general property and then for a very special case of that general property.

Example 6 on page 112 is incorrect, as the sets H and K described there have more than T in common. This example may be corrected as follows: For each n and i, $(1 \leqq i \leqq 2^{n-1})$, let B_{in} denote the point $((2i-1)/2^n, 1/n)$, and let C_{in} denote the point $((2i-1)/2^n, 0)$. Then with T as defined in the book, let H be T plus all the intervals $B_{in}C_{in}$ for i odd, and let K be T plus all such intervals for i even. The sets as thus constructed have the desired properties.

Theorem 90, page 271, is incorrect unless the fourth line be changed to read: ". . . mutually separated *connected* point sets . . . ".

Theorem 8, page 435, is partly false, as is shown by the author's own statement that a segment of an arc is topologically equivalent to an open curve. An open curve is a continuum, while a segment is connected but not closed. Hence the collection of all closed sets and the collection of all continua are not topological collections, and these statements should be deleted from the theorem.

Corrections to Theorems 6, page 434, and 19, page 451, are indicated below.

There are numerous misprints throughout the book, most of which are easily discovered and corrected by one making a careful study of the book. The following errors are worthy of noting here. On page 345, line 9, replace G by H. On page 352, line 2, replace the second D by U, and in line 3, replace k by n.

The usefulness of the book for reference purposes is marred by the fact that a number of definitions have been omitted from the glossary. Among them are: outer boundary (p. 193), Property A (p. 284), Axiom 6 (p. 368).

Certain definitions have been misplaced in the text. The definition of nondegenerate (p. 47) should precede the definition of totally disconnected (p. 30). The U^* notation is used many times before it is formally defined on page 110.

Usually its meaning is given, but such is not the cases on pages 91 and 92. Continuum of condensation is mentioned on page 76 but is not defined until page 106.

A different type of misplacement occurs in the case of the definitions of simple closed surface and cactoid. These are defined on page 151, but are not mentioned further until much later in the book.

Other definitions seem to have been omitted entirely from the book, such as simple domain and contracting sequence (p. 453).

There is a possibility of confusion on page 452 ff., because it is nowhere shown that a simple closed surface is a number sphere. In Theorems 20 and 21, S' is called a *space*; in the following theorems it is a *simple closed surface*, but the author has not shown that this change is justified.

But the reader of the book will probably be more concerned by the fact that the author does not discuss the significance of the axioms nor the logical interrelations between them. Questions such as the following will naturally occur to the reader, and we shall attempt to answer these questions below. (1) Why is Axiom 0 assumed in Chapter 1, but not thereafter? (2) There are two different propositions called Axiom 6. Are they logically equivalent? (3) The notation for Axioms 5, 5_1, 5_2 suggests some relation between these axioms. Just what is this relation? (4) Are there any further logical relations between the numbered axioms?

(1) The answer to this question is that the author has tacitly assumed Axiom 0 throughout the book and there is therefore no reason for including it in the title of Chapter 1 and omitting it from the titles of the following chapters.

For if we do not assume Axiom 0 to be true, we are confronted by the peculiar situation that a region is not necessarily a point set, but that all the definitions of Chapter 1 are worded on the assumption that a region *is* a point set. If in Chapter 2, we do not assume Axiom 0, our domain of discourse is the set of all points S and the set of all regions T. By Axiom 1, certain subsets of T *cover* S. But *cover* has been defined on page 5 for collections of point sets only.

Let us first suppose that it is permissible to define *cover* arbitrarily for collections which are not collections of point sets. It can then be shown that with the proper definition of *cover*, certain theorems of Chapter 2 are false. Hence not all the theorems of Chapter 2 follow from Axioms 1 and 2 alone, if *cover* be defined arbitrarily.

Let us then define *cover* thus: The collection G of sets is said to *cover* the point set M if each point of M belongs to some set of the collection G. With this definition, for any set X, let us denote by X' the point set consisting of all points which belong to the set X. Then if a point set M is covered by a collection G of sets (X), it is also covered by the collection G' of point sets (X'), corresponding to the sets (X) of G. Axiom 1 shows the existence of collections of regions for which not all of the sets (X') are vacuous. If we replace *region* R wherever it occurs by *point set* R' *consisting of all points contained in the region* R, all the statements of the book (including those of Chapter 1) are true whether Axiom 0 be assumed or not.

Rather than make such a replacement, it seems simpler to assume explicitly Axiom 0 throughout the book, and we shall do so hereafter. In that case

Axiom 0 should be included with the other axioms in the titles of Chapter 2 and the following chapters.

(2) It is easy to construct examples to show that Axiom 6 on page 368 is a weaker form of Axiom 6 on page 412 and that the theorem on page 434 is not true if the weaker form of Axiom 6 is used. Since the weaker form of Axiom 6 is used only in connection with the second part of Chapter 5, the easiest way to correct this matter is to denote the axiom on page 368 by 6_0 instead of 6. It might be noted again that Axiom 6_0 was overlooked in the construction of the glossary.

(3) Concerning Axioms 5, 5_1, 5_2, the following facts may be of interest. If Axioms 0, 1_0, 3 are true, then no region (and hence no domain) is degenerate. If Axioms 0, 1_0, 3, 5_1 are true, then Axiom 2 is true.

The title of Chapter 6 is *Consequences of Axioms* 1, 2, 4, 5_1, 5_2, 6, 7. However, in the proof of Theorem 2, a theorem is used whose proof seems to depend on Axiom 3. Thus the author seems to be assuming Axiom 3 also (and Axiom 0 of course), and in that case Axiom 2 may be omitted from the list of assumptions, as we have just shown. Hence it seems that the proper title for this chapter is *Consequences of Axioms* 0, 1, 3, 4, 5_1, 5_2, 6, 7.

From Theorems 2 and 3 of this chapter it follows that Axiom 5 is a logical consequence of Axioms 0, 1, 3, 4, 5_1, 5_2. This is a partial explanation of the notation used for these axioms.

(4) From the logical relations pointed out in the preceding section, we see that the hypothesis of Theorem 6, page 434, should read: ". . . Axioms 0, 1, 3, etc."

Since every subset of a compact set is compact, whenever Axiom 8′ is true, then Axiom 6 is true. Hence the hypothesis of Theorem 19, page 451, should read: ". . . Axioms 0, 1, 3, 4, 5_1, 5_2, 7, 8′ . . .".

In conclusion, the reviewer wishes to state that this book will undoubtedly prove to be an excellent text from which to obtain an insight into the nature of the problems considered by the school of mathematicians headed by Professor Moore. The book is a worthy addition to the set of volumes that have been published by this Society.

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