THREE BOOKS ON TOPOLOGY

Einfachste Grundbegriffe der Topologie. By Paul Alexandroff, with an introduction by David Hilbert. Berlin, Julius Springer, 1932. 48 pp.

Einführung in die kombinatorische Topologie. By Kurt Reidemeister. Braunschweig, Vieweg, 1932. xii+ 209 pp.

Knotentheorie (vol. 1 of the Ergebnisse der Mathematik und ihrer Grenzgebiete). By K. Reidemeister. Berlin, Julius Springer, 1932. vi+114 pp. and 114 figures.

To the uninitiate in matters of combinatorial analysis situs the first two of these books on topology will seem hardly to deal with the same subject, so completely do they complement each other. The little book of Alexandroff is an admirable and simple exposition of the ideas centering around homology and dimension, while Reidemeister's book, more complete and more important scientifically, is concerned with the study of abstract discrete groups and their topological applications.

Alexandroff has written particularly for those who do not care to undertake a systematic study of topology, and his enthusiasm in exhibiting the beauties of the subject will not escape the reader. There are many simple examples and the accompanying drawings are so skillfully made that one can not fail to see how the homology group operates, or to appreciate its intuitive meaning. The author does not allow the proofs of important matters to depend on the reader's intuition; on the contrary, it is remarkable how much he has treated in complete scientific detail. There is for example a very readable proof of the invariance of the Betti numbers (modeled after an elegant proof by Alexander) and a proof of the author's own important deformation theorem, with a corollary in the theory of dimensionality. The reader who wishes to learn more about these subjects will know where to begin from the numerous references to other authors; but for those who do not intend to consult the original papers, the references will not always give a complete nor accurate impression of the history of the subject. For example, the first formulation of the very important "Pflastersatz" was by Lebesgue in 1911, a fact that might well have been mentioned, despite the incompleteness of Lebesgue's proof. And again, what the author presumably means by the Lefschetz-Hopf fixed-point formula is surely due to Lefschetz, and should be named, we believe, accordingly.

For the purposes of his book, the author was undoubtedly wise in limiting himself to the homology theory; as for its importance, the author points out that it is this part of topology which will tend more and more to govern the development of the whole subject. This is undoubtedly true in the *unified* theory of general spaces, to which the author has made distinguished contributions. On the other hand there remains the fundamental problem of classifying manifolds, before which the homology apparatus, by itself, seems powerless; and even in the transformation theory which the author mentions, these methods seem to yield only broad variations of one special type of result. In these and other questions, one encounters not only the homology groups, which are abelian and therefore relatively tractable, but also the much more unruly Poincaré groups which are defined by discrete sets of generators and finite sets of generating relations. One realizes the difficulties involved in the study of such groups from the fact that as yet there is no known general method of determining whether or not two symbols represent the same group element (the word-problem).

In recent years the works of Dehn, Nielsen, Reidemeister, Schreier, and others have brought about something like the beginnings of a general theory of discrete groups, and it is precisely to this theory that Reidemeister's *Einführung* is devoted. The first half of the book is purely group-theoretical. Here we find solutions of a number of special cases of the word-problem, and a number of results concerning the generation of subgroups, with Schreier's well known theorem, that any subgroup of a free group is free, coming out in various ways as a corollary. The remainder of the book deals with applications to one- and two-dimensional complexes and their covering complexes. This is the topological part, and the author has succeeded in making the treatment abstract throughout, in the sense that no limiting process is ever used or implied. An abstract reduction of surfaces to normal forms appears to be new and here the dual manifold turns out to be very useful. These parts are not easy to read. It seems to the reviewer that a few figures (there is not one) would have helped enormously, without compromising the abstractness of the treatment. It would have helped for example to see what the dual of a manifold which only contains one 2-cell really looks like; and in the definition of the representative graph (Gruppenbild) of a group, an example or two would have made it unnecessary for one reader, at least, to consult the original paper of Dehn as an introduction to the *Einführung*.

In his *Knotentheorie* Reidemeister is less severe with the reader, and various types of knot diagrams and operations on them are nicely illustrated. There are a number of ways of associating a matrix or a group with a knot, and the problem of classifying knots then becomes one of finding such calculable invariants of the groups and matrices as are also *knot invariants*. Practically all known results along these lines, including several hitherto unpublished, are included in this work, and the author's own contributions in which the interrelations of the various groups are developed serve effectively to unite the existing theory. The knot problem is perhaps the oldest problem in analysis situs. Gauss defined looping coefficients, and Listing, in the first published *Topologie* of mathematics, devoted much space to knots. Later, the knot problem assumed a temporary importance in physics. An atom was thought of as a knotted vortex in the ether and it was natural to suppose that the properties of an element depended on the knot structure of its atoms. More recently, it has been pointed out that the problem of classifying three-dimensional manifolds may be intimately connected with that of classifying knots. In one of Tait's classic papers on knots (1879), he admits that he "has not succeeded in catching the right note." Today we have a "theory" and have built much ingenious mathematics around the concept of knot. But what more do we actually know about knots themselves? The answer seems to be this: any knot whose plane projection contains nine or fewer double points can be recognized as belonging to one of a finite number of non-equivalent types. This meager knowledge depends partly on the empirical results of Tait and his contemporaries, partly on the known calculable knot invariants such as those discovered by Alexander, and partly on special results (some of which are still unpublished) concerning pairs of knots which are not distinguishable by their invariants. Thus the problem is still open and still fascinating—the more so since it is now apparent that even though the problem seems to be one of abstract groups, progress may depend on the results of the most unexpected domains of algebra. The complete and concise little work of Reidemeister will do much to encourage further attacks.

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MOORE ON POINT SETS

Foundations of Point Set Theory. By R. L. Moore. American Mathematical Society Colloquium Publications, Volume 13. New York, 1932. viii +486 pp.

We are told in the Preface that this volume is intended to be a self-contained treatment of the foundations of the point-set-theoretic branch of analysis situs. It is concerned chiefly with those topics which are the results of Professor Moore's own research. Hence the book does not mention certain topics—dimension theory, for instance—which are closely connected with the topics discussed, but with the development of which Professor Moore was not primarily concerned.

The present treatment of point set theory is based upon a system of axioms, the undefined notions being *point* and *region*. General logical concepts are assumed, including in particular those of the logic of classes and the fundamental propositions concerning integers. Of these concepts, the author states explicitly only the Zermelo Axiom, which is to be considered as included among his assumptions. In addition to these introductory statements, the Introduction (pp. 1–4) is concerned with definitions of various types of sequences.

Chapter 1 (pp. 5–85) contains the theorems which can be derived from Axioms 0 and 1. The author also shows that most of these theorems follow from Axioms 0 and 1_0, where 1_0 is a weaker form of Axiom 1. Since Axioms 1 and 1_0 each consist of several parts, it is not surprising that so many theorems can be proved from them. While certain theorems can be proved with still weaker hypotheses, the author makes no attempt to do so. But the group of examples on pages 24–28 shows that in the case of certain theorems at least, no unnecessary restrictions have been placed in the hypotheses to compensate for the weakness of the assumptions concerning the space containing the sets. It is shown by these examples that under weaker hypothese the conclusions of this group of theorems become false, even if we assume that the underlying space is the euclidean plane.

The topics discussed in Chapter 1 are concerned with the following ideas, arranged roughly in order: boundary point, sequential limit point, Bo el property, connectivity, irreducible continuum, limiting set, separation of two sets