CONVERGENCE FACTORS FOR DOUBLE SERIES*

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1. *Introduction*. By a theorem due originally to Frobenius† if the power series $y(z)=\sum_{i=0}^{\infty}a_iz^i$ has the unit circle as circle of convergence, and if $\sum_{i=0}^{\infty}a_i$ is summable by Cesàro's first mean with the value s, then $\lim y(z)=s$ as $z\rightarrow+1$ along any path lying between two fixed chords intersecting at $z=+1$. This theorem has been considerably extended, in the field of double series notably by Bromwich and Hardy,‡ and by C. N. Moore.§ The former proved that if $f(x, y)=\sum_{i,j=0}^{\infty}a_{ij}x^iy^j$, and if $|S_{ij}^{(k)}|$, the kth Hölder mean of $\sum a_{ij}$, is bounded for all values of i and j, and $\lim_{i,j\rightarrow\infty} S_{ij}^{(k)}=s$, then also $\lim_{x,y\rightarrow 1} f(x, y)=s$. More particular reference will presently be made to Moore's paper, his theorems being the starting point for the present article. Robison,‖ also, has given necessary and sufficient conditions for the regularity of a transformation applied to a double sequence.

The writer, in a paper on series of the form $y(z)=\sum_{i=0}^{\infty}a_iz^{f(i)}$, gave sufficient conditions on $f(i)$ so that $\lim_{z\rightarrow 1} y(z)=s$.¶ The present paper deals with double series of the type

$$J(z, w) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}z^{f(i)}w^{g(j)},$$

where z, w are complex variables, and $f(i)$, $g(j)$ are logarithmico-exponential functions,** called for brevity L-functions. Sufficient conditions on $f(i)$, $g(j)$ will be given so that if $\sum a_{ij}$ is summable $(C, r-1)$ with the value s, then $J(z, w)$ will be convergent for $|z|<1$, $|w|<1$, and $\lim_{(z,w)\rightarrow(1,1)} J(z, w)=s$.

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* Presented to the Society, April 8, 1932.

† Journal für Mathematik, vol. 89 (1880), p. 262.

‡ Proceedings of the London Mathematical Society, (2), vol. 2 (1904), p. 161.

§ Transactions of this Society, vol. 29 (1927), p. 227.

‖ Transactions of this Society, vol. 28 (1926), p. 50.

¶ American Journal of Mathematics, vol. 53 (1931), p. 817.

** Hardy, *Orders of Infinity*.

2. *Notation.* We shall employ Moore's notation. Thus

$$(1)\quad s_{m_1m_2} = \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} a_{ij},$$

$$(2)\quad S_{m_1m_2}^{(k)} = \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} \frac{\Gamma(k+m_1-i)}{\Gamma(k)\cdot\Gamma(m_1-i+1)}\cdot\frac{\Gamma(k+m_2-j)}{\Gamma(k)\cdot\Gamma(m_2-j+1)} s_{ij},$$

$$(3)\quad A_{m_1m_2}^{(k)} = \frac{\Gamma(m_1+k)}{\Gamma(k+1)\cdot\Gamma(m_1)}\cdot\frac{\Gamma(m_2+k)}{\Gamma(k+1)\cdot\Gamma(m_2)}.$$

If the quotient $S_{m_1m_2}^{(k)}/A_{m_1m_2}^{(k)}$ approaches a limit s as m_1, m_2 become infinite, we say that the series $\sum a_{ij}$ is summable (C, k) with the value s. We shall also have occasion to employ the following notation:

$$(4)\quad \phi_{ij}(z, w) = z^{f(i)}w^{g(j)},$$

$$(5)\quad \phi_{ij}^{(p,q)}(z, w) = \frac{\partial^{p+q}\phi_{ij}(z, w)}{\partial i^p \partial j^q},$$

$$(6)\quad \Delta_{rr}\phi_{ij}(z, w) = \sum_{s_1=0}^{r}\sum_{s_2=0}^{r}(-1)^{s_1}(-1)^{s_2}\binom{r}{s_1}\binom{r}{s_2}\phi_{i+s_1,j+s_2}(z,w),$$

$$(7)\quad \Delta_{r0}\phi_{ij}(z, w) = \sum_{s_1=0}^{r}(-1)^{s_1}\binom{r}{s_1}\phi_{i+s_1,j}(z, w).$$

The region within which $|z| < 1$, $|w| < 1$, will be denoted by $E(z, w)$, and the open region in the neighborhood of (1, 1) lying between two chords of the unit circle intersecting at $+1$, by $E'(z, w)$.

THEOREM. *If $\sum a_{ij}$ is summable $(C, r-1)$ with the value s, when $r \geqq 1$ is an integer, and if*

(a) $|S_{ij}^{(r-1)}/A_{ij}^{(r-1)}| < C$, *$(i, j = 1, 2, \cdots;$ C a constant);*

(b) *$f(t)$, $g(t)$ are L-functions which, together with their first $(r-1)$ derivatives, exist and are continuous, are of constant sign, and are monotonic for $t \geqq 1$;*

(c) $\log t = \sigma[f(t)]$, $\log t = \sigma[g(t)]$;

(d) *$f(t) = \sigma(t^\alpha)$, $g(t) = \sigma(t^\alpha)$ for some $\alpha > 0$;*

then the double series $J(z, w) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}z^{f(i)}w^{g(j)}$ will converge in $E(z, w)$, and $\lim_{(z,w)\to(1,1)} J(z, w) = s$, the paths of approach lying in $E'(z, w)$.

3. *Statement of Lemmas.* For the sake of brevity the following lemmas are stated here without proof.*

LEMMA 1. *As* $(z, w)\to(1, 1)$ *in* $E'(z, w)$, $|\log z| = O[\log \rho]$, $|\log w| = O(\log \tau)$, *where* $\rho = |z|$, $\tau = |w|$.

LEMMA 2. *If* $h(t)$ *satisfies conditions* (c) *and* (d) *of the Theorem, then for* $k \geqq 1$, $h^{(k)}(t)/h(t) = O(1/t^k)$, *and* $h^{(k)}(t)/h'(t) = O(1/t^{k-1})$.

4. *General Relations.* Each term $T[\phi_{ij}^{(p,q)}(\rho,\tau)]$ of $\phi_{ij}^{(p,q)}(\rho,\tau)$ is of the form

$$(8)\qquad B_1\rho^{f(i)}\tau^{g(j)}\prod_{\lambda=1}^{p}[f^{(\lambda)}(i)]^{\alpha_\lambda}\prod_{\sigma=1}^{q}[g^{(\sigma)}(j)]^{\beta_\sigma}(\log\rho)^\alpha(\log\tau)^\beta,$$

where B_1 is a constant, $\alpha=\sum_{\lambda=1}^{p}\alpha_\lambda$, $p=\sum_{\lambda=1}^{p}\lambda\alpha_\lambda$, $\beta=\sum_{\sigma=1}^{q}\beta_\sigma$, $q=\sum_{\sigma=1}^{q}\sigma\beta_\sigma$, and any, or all but one, of α_λ or β_σ may be zero. It will be noted that $p\geqq\alpha$, $q\geqq\beta$. By Lemma 2 we have

$$(9)\quad |T[\phi_{ij}^{(p,q)}(\rho,\tau)]| \leqq B_2\rho^{f(i)}\tau^{g(j)}\frac{[f(i)]^\alpha[g(j)]^\beta}{i^pj^q}|\log\rho|^\alpha|\log\tau|^\beta,$$

$$(10)\quad |T[\phi_{ij}^{(p,q)}(\rho,\tau)]| \leqq B_3\rho^{f(i)}\tau^{g(j)}\frac{[f'(i)]^\alpha[g'(j)]^\beta}{i^{p-\alpha}j^{q-\beta}}|\log\rho|^\alpha|\log\tau|^\beta.$$

By Lemma 1, in $E'(z, w)$, $|\phi_{ij}^{(p,q)}(z, w)| \leqq B_4|\phi_{ij}^{(p,q)}(\rho,\tau)|$; so that if we denote by $\sum T_{ij}^{(p,q)}(\rho,\tau)$ the sum of all terms of $\phi_{ij}^{(p,q)}(\rho,\tau)$ whose signs are unlike that of $\rho^{f(i)}\tau^{g(j)}[f(i)\log\rho]^p[g(j)\log\tau]^q$, the leading term, we have

$$(11)\qquad \Phi \leqq (-1)^p(-1)^qB_4\{\phi_{ij}^{(p,q)}(\rho,\tau) - 2\sum T_{ij}^{(p,q)}(\rho,\tau)\},$$

where $\Phi = |\phi_{ij}^{(p,q)}(z, w)|$. From (6) and (11) we obtain

$$|\Delta_{rr}\phi_{ij}(z, w)| \leqq B_4\int_0^1 d\xi_1\int_0^1 d\eta_1\cdots$$

$$(12)\qquad \int_0^1 d\xi_r\int_0^1\{\phi_{\mu\nu}^{(r,r)}(\rho,\tau) - 2\sum T_{\mu\nu}^{(r,r)}(\rho,\tau)\}d\eta_r,$$

where $\mu=i+\xi_1+\cdots+\xi_r$, $\nu=j+\eta_1+\cdots+\eta_r$. By (9), since $i\leqq\mu$, $\rho^{f(u)}\leqq\rho^{f(i)}$, $f(\mu)\leqq f(i+r)$, with similar inequalities for j and ν, we have for fixed (z, w), if we set $M = |\Delta_{rr}\phi_{ij}(z, w)|$,

* The proof of Lemma 1 may be found in my paper cited above; Lemma 2 may be deduced from certain remarks by Hardy, in his *Orders of Infinity*.

$$
\begin{aligned}
M &\leqq B_5 \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \int_0^1 d\xi_r \int_0^1 \rho^{f(\mu)} \tau^{g(\nu)} \frac{[f(\mu)]^\alpha [g(\nu)]^\beta}{\mu^r \nu^r} d\eta_r \\
&\leqq B_5 \rho^{f(i)} \tau^{g(j)} \frac{[f(i+r)]^r [g(j+r)]^r}{i^r j^r}.
\end{aligned} \tag{13}
$$

It can be shown easily that if a, b are positive constants

$$
\lim_{i\to\infty} \rho^{f(i)} [f(i+a)]^b = \lim_{j\to\infty} \tau^{g(j)} [g(j+a)]^b = 0; \tag{14}
$$

whence

$$
\begin{aligned}
\lim_{i,j\to\infty} \left| \Delta_{rr}\phi_{ij}(z, w) \right| &= \lim_{i\to\infty} \left| \Delta_{rr}\phi_{ij}(z, w) \right| \\
&= \lim_{j\to\infty} \left| \Delta_{rr}\phi_{ij}(z, w) \right| = 0.
\end{aligned} \tag{15}
$$

5. *Proof of Theorem*. C. N. Moore* has given necessary and sufficient conditions that a double series $\sum a_{ij}F_{ij}(z, w)$ shall converge in $E(z, w)$ and approach a limit s as $(z, w)\to(1, 1)$ in $E'(z, w)$, the series $\sum a_{ij}$ being summable $(C, r-1)$ with the value s, and condition (a) of the Theorem being satisfied. For series of our type, $\sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} z^{f(i)} w^{g(j)}$, these conditions are:

(A) $\displaystyle\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} i^{r-1} j^{r-1} \left| \Delta_{rr}\phi_{ij}(z, w) \right| < K(z, w), \qquad (E(z, w));$

(B₁) $\displaystyle\lim_{j\to\infty} j^{r-1} \sum_{i=1}^{p} i^{r-1} \left| \Delta_{r0}\phi_{ij}(z, w) \right| = 0, \quad (E(z, w);\ p = 1, 2, \cdots);$

(B₂) $\displaystyle\lim_{i\to\infty} i^{r-1} \sum_{j=1}^{q} j^{r-1} \left| \Delta_{0r}\phi_{ij}(z, w) \right| = 0, \quad (E(z, w);\ q = 1, 2, \cdots);$

(C) $i^{r-1} j^{r-1} \left| \phi_{ij}(z, w) \right| < M(z, w), \quad (E(z, w);\ i, j = 1, 2, \cdots);$

(A′) $\displaystyle\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} i^{r-1} j^{r-1} \left| \Delta_{rr}\phi_{ij}(z, w) \right| < K, \qquad (E'(z, w));$

(D₁) $\displaystyle\lim_{(z, w)\to(1, 1)} \sum_{j=q}^{\infty} j^{r-1} \left| \Delta_{rr}\phi_{ij}(z, w) \right| = 0, \qquad (i, q = 1, 2, \cdots);$

(D₂) $\displaystyle\lim_{(z, w)\to(1, 1)} \sum_{i=p}^{\infty} i^{r-1} \left| \Delta_{rr}\phi_{ij}(z, w) \right| = 0, \qquad (p, j = 1, 2, \cdots);$

(E) $\displaystyle\lim_{(z, w)\to(1, 1)} \phi_{ij}(z, w) = 1, \qquad (i, j = 1, 2, \cdots);$

* Loc. cit.

where $K(z, w)$ and $M(z, w)$ are finite for each (z, w) in $E(z, w)$, and K is a positive constant. We proceed to show that these eight conditions are fulfilled in the present case.

CONDITION (A). By (12), since $i \leqq \mu, j \leqq \nu$,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} i^{r-1} j^{r-1} \left| \Delta_{rr}\phi_{ij}(z, w)\right| \leqq B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \tag{16}$$
$$\cdot \int_0^1 d\xi_r \int_0^1 \mu^{r-1}\nu^{r-1}\left\{\phi_{\mu\nu}^{(r,r)}(\rho, \tau) - 2\sum T_{\mu\nu}^{(r,r)}(\rho, \tau)\right\} d\eta_r .$$

Considering first that part of this integrand involving $\phi_{\mu\nu}^{(r,r)}$, and integrating by parts with respect to η_r and then with respect to ξ_r, we obtain

$$B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \int_0^1 d\xi_r \int_0^1 \mu^{r-1}\nu^{r-1}\phi_{\mu\nu}^{(r,r)}(\rho, \tau) d\eta_r$$
$$= B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left\{G(i,j) - G(i+1,j) - G(i,j+1) + G(i+1,j+1)\right\}, \tag{17}$$

where

$$G(i,j) = \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \int_0^1 d\xi_{r-1} \int_0^1 \sum_{s=0}^{r-1}\sum_{t=0}^{r-1} (-1)^s(-1)^t \tag{18}$$
$$\cdot \frac{(r-1)!}{(r-1-s)!}\,\frac{(r-1)!}{(r-1-t)!}\,\mu_0^{r-1-s}\,\nu_0^{r-1-t}\,\phi_{\mu_0\nu_0}^{(r-1-s,\,r-1-t)}(\rho, \tau) d\eta_{r-1},$$

in which expression $\mu_0 = i + \xi_1 + \cdots + \xi_{r-1}$, and $\nu_0 = j + \eta_1 + \cdots + \eta_{r-1}$. By the aid of (9) we find

$$G(i,j) \leqq B_6[(r-1)!]^2 \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots$$
$$\cdot \int_0^1 d\xi_{r-1} \int_0^1 \sum_{s=0}^{r-1}\sum_{t=0}^{r-1} \rho^{f(\mu_0)}\tau^{g(\nu_0)}\left[f(\mu_0)\right]^{r-1-s}\left[g(\nu_0)\right]^{r-1-t} d\eta_{r-1} \tag{19}$$
$$\leqq B_6(r!)^2\rho^{f(i)}\tau^{g(j)}\left[f(i+r-1)\cdot g(j+r-1)\right]^{r-1}.$$

This expression, by virtue of (14), approaches zero when i, or j, or both, increase indefinitely; so that

$$B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\{G(i,j) - G(i+1,j) - G(i,j+1) + G(i+1,j+1)\right\}$$
$$= B_4 G(1,1) \leqq B_7(r!)^2\rho^{f(1)}\tau^{g(1)}\left[f(r)\cdot g(r)\right]^{r-1}. \tag{20}$$

Thus (17) is bounded for fixed z and w.

Returning now to the remaining part of the integrand in (16), we have, by (10), for each term

$$
(21)\quad \begin{aligned}
&2B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \int_0^1 d\xi_r \int_0^1 -\mu^{r-1}\nu^{r-1}T_{\mu\nu}^{(r,r)}(\rho,\tau)d\eta_r \\
&\qquad \leqq 2B_4 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \\
&\cdot \int_0^1 d\xi_r \int_0^1 B_3 \mu^{\alpha-1}\nu^{\beta-1}\rho^{f(\mu)}\tau^{g(\nu)}[f'(\mu)]^{\alpha}[g'(\nu)]^{\beta} \left| \log \rho \right|^{\alpha} \left|\log \tau\right|^{\beta} d\eta_r.
\end{aligned}
$$

This integrand is, except for a constant factor, the leading term of $\mu^{\alpha-1}\nu^{\beta-1}\phi_{\mu\nu}^{(\alpha,\beta)}(\rho, \tau)$. Now $T_{\mu\nu}^{(r,r)}(\rho, \tau)$, being negative, cannot be the leading term of $\phi_{\mu\nu}^{(r,r)}$; hence $\phi_{\mu\nu}^{(\alpha,\beta)}$ is of lower order than $\phi_{\mu\nu}^{(r,r)}$, and may be substituted for $T_{\mu\nu}^{(r,r)}$. We now set up a new expression, like (16) but with $\phi_{\mu\nu}^{(\alpha,\beta)}$ in place of $\phi_{\mu\nu}^{(r,r)}$, and sufficient repetition of this process must eventually lead, by (20), to

$$
(22)\quad \begin{aligned}
&B_8 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^1 d\xi_1 \int_0^1 d\eta_1 \cdots \\
&\qquad \cdot \int_0^1 d\xi_r \int_0^1 \phi_{\nu,\mu}^{(1,1)}(\rho, \tau) d\eta_r \leqq B_9 \rho^{f(1)}\tau^{g(1)}.
\end{aligned}
$$

We have, therefore,

$$
(23)\quad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{r-1}j^{r-1}\left|\Delta_{rr}\phi_{ij}(z, w)\right| < B_{10}(r!)^2\rho^{f(1)}\tau^{g(1)}[f(r)\cdot g(r)]^{r-1},
$$

which proves that condition (A) is satisfied. By an entirely similar procedure we find that

$$
(24)\quad \begin{aligned}
&j^{r-1}\sum_{i=1}^{p} i^{r-1}\left| \Delta_{r0}\phi_{ij}(z, w)\right| \\
&\quad \leqq B_{11}j^{r-1} \sum_{i=1}^{p} \int_0^1 d\xi_1 \int_0^1 d\xi_2 \cdots \int_0^1 d\xi_{r-1} \int_0^1 \mu^{r-1} \\
&\quad \cdot \left\{ \phi_{\mu j}^{(r,0)}(\rho, \tau) - 2 \sum T_{\mu j}^{(r,0)}(\rho, \tau)\right\} d\xi_r \\
&\quad < B_{12} j^{(r-1)}(r!)\tau^{g(j)}\left\{\rho^{f(p+1)}[f(p+r)]^{r-1} - \rho^{f(1)}[f(r)]^{r-1}\right\}.
\end{aligned}
$$

By (14) $\lim_{p\to\infty} \rho^{f(p+1)}[f(p+r)]^{r-1}=0$, so that the expression within the braces is bounded for $p\geqq 1$; and since, by condition (c) of the Theorem, $\lim_{j\to\infty} j^{r-1}\tau^{g(j)}=0$, we have

$$\lim_{j\to\infty} j^{r-1}\sum_{i=1}^{p} i^{r-1}\left|\Delta_{r0}\phi_{ij}(z, w)\right| = 0, \quad (E(z, w);\ p = 1, 2, \cdots).$$

Condition (B_1) is therefore satisfied. The argument for (B_2) is, of course, precisely similar.

Proceeding to condition (C), we note that by condition (c) of the Theorem, for an assigned $\epsilon>0$ there exist i_0, j_0, such that for $i>i_0$, $\log i<\epsilon f(i)$, and for $j>j_0$, $\log j<\epsilon g(j)$. By choosing ϵ less than both $|\log \rho|/(r-1)$, and $|\log \tau|/(r-1)$ we have, for such i and j, $\epsilon(r-1)f(i)<f(i)|\log \rho|$, $\epsilon(r-1)g(j)<g(j)|\log \tau|$, and hence

$$(25)\qquad \begin{aligned} i^{r-1}j^{r-1}\left|\phi_{ij}(z, w)\right| &= i^{r-1}j^{r-1}\rho^{f(i)}\tau^{g(j)} \\ &< e^{(r-1)\{\log i-\epsilon f(i)\}}\cdot e^{(r-1)\{\log j-\epsilon g(j)\}} < 1. \end{aligned}$$

Condition (C) is therefore satisfied.

In condition (A), the bound $K(z, w)$ depends upon z and w, for the constant B_{10} in (23) depends upon $\log \rho$ and $\log \tau$. We now further define $E'(z, w)$ as follows. For a given L, $0<L<1$, let all values of (z, w) in $E'(z, w)$ be such that $|\log \rho| \leqq |\log L|$, $|\log \tau| \leqq |\log L|$. If we now set B_{13} equal to the value of B_{10} corresponding to $\rho=\tau=L$, we have, for all (z, w) in $E'(z, w)$,

$$(26)\qquad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^{r-1}j^{r-1}\left|\Delta_{rr}\phi_{ij}(z, w)\right| < B_{13}(r!)^2[f(r)\cdot g(r)]^{r-1} = K.$$

Thus condition (A′) is satisfied.

For condition (D_1) we have by (6), for fixed i,

$$(27)\qquad \begin{aligned} &\sum_{j=q}^{\infty} j^{r-1}\left|\Delta_{rr}\phi_{ij}(z, w)\right| \\ &= \sum_{j=q}^{\infty} j^{r-1}\left|\sum_{s_1=0}^{r}\sum_{s_2=0}^{r}(-1)^{s_1}(-1)^{s_2}\binom{r}{s_1}\binom{r}{s_2}\phi_{i+s_1,\, j+s_2}(z, w)\right| \\ &\leqq \left|\sum_{s_1=0}^{r}(-1)^{s_1}\binom{r}{s_1}z^{f(i+s_1)}\right|\sum_{j=q}^{\infty} j^{r-1}\left|\sum_{s_2=0}^{r}(-1)^{s_2}\binom{r}{s_2}w^{g(j+s_2)}\right|. \end{aligned}$$

The first part of this expression is the sum of $(r+1)$ terms, each a continuous function of z; whence

$$(28) \qquad \lim_{s\to 1} \left| \sum_{s_1=0}^{r} (-1)^{s_1} \binom{r}{s_1} z^{f(i+s_1)} \right| = 0.$$

Next, if $z \neq 0$, we have by (7),

$$(29) \quad \sum_{j=q}^{\infty} j^{r-1} \left| \sum_{s_2=0}^{r} (-1)^{s_2} \binom{r}{s_2} w^{g(j+s_2)} \right| = \frac{1}{\rho^{f(i)}} \sum_{j=q}^{\infty} j^{r-1} \left| \Delta_{0r}\phi_{ij}(z, w) \right| .$$

By a procedure similar to that followed for (B_1) we now find

$$(30) \quad \frac{1}{\rho^{f(i)}} \sum_{j=q}^{\infty} j^{r-1} \left| \Delta_{0r}\phi_{ij}(z, w) \right| < B_{14}(r!)\tau^{g(q)}[g(q + r - 1)]^{r-1}.$$

This is bounded for $\tau < 1$, and fixed q. Therefore

$$(31) \qquad \lim_{(z,w)\to(1,1)} \sum_{j=q}^{\infty} j^{r-1} \left| \Delta_{rr}\phi_{ij}(z, w) \right| = 0,$$

and condition (D_1) is satisfied. The argument for (D_2) is exactly similar.

Finally, for condition (E) we have

$$(32) \qquad \lim_{(z,w)\to(1,1)} \phi_{ij}(z, w) = \lim_{(z,w)\to(1,1)} z^{f(i)} w^{g(j)} = 1;$$

and this completes the proof of the Theorem.

It will be noted that condition (c), $\log t = \sigma[f(t)]$, etc., is necessitated by Moore's condition (C), $i^{r-1}j^{r-1}|\phi_{ij}(z, w)| < M(z, w)$. It insures the convergence of the series $J(z, w)$. If, however, a suitable restriction be placed upon $s_{m_1 m_2}$, namely, $s_{m_1, m_2} = O[\lambda^{f(m_1)+g(m_2)}]$ for every $\lambda > 1$, condition (c) may be omitted. We may then have $f(t) = \log t$, $g(t) = \log t$, or even more slowly increasing functions. The proof, however, is somewhat long. It will be observed that the Theorem can be extended in an obvious way to multiple series of order n.

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