

BLOCH'S THEOREM FOR MINIMAL SURFACES*

BY E. F. BECKENBACH†

The following theorem was first proved by Bloch.‡

BLOCH'S THEOREM. *There exists a positive absolute constant B with the following property. Let $Z=f(z)$ be analytic for $|z| \leq 1$, with $|f'(0)| \geq 1$; then in the Z -plane there is an open circle of radius at least B , which is the uniplanar§ map of a portion of the circle $|z| < 1$.*

Other proofs of greater simplicity have been given. The present generalization follows the proof given by Landau and by Valiron.|| In this paper we shall prove the following theorem.

THEOREM. *There exists a positive absolute constant B with the following property. Let the circle $u^2+v^2 \leq 1$ be mapped conformally on a minimal surface, with $\mathcal{E}_0 \geq 1$, where \mathcal{E}_0 denotes the area deformation ratio at the origin; then on the minimal surface there is an open geodesic circle of radius at least B , containing no singular points, which is the one-to-one map of a portion of the circle $u^2+v^2 < 1$.*

That is, there is a point on the surface such that no curve on the surface, issuing from this point and of length less than B , comes either to the boundary of the map or to a point where the conformal character of the map breaks down.

In order that the real analytic functions

$$x_j = x_j(u, v), \quad (j = 1, 2, 3),$$

* Presented to the Society, February 25, 1933.

† National Research Fellow.

‡ *Les théorèmes de M. Valiron sur les fonctions entières, et la théorie de l'uniformisation*, Comptes Rendus, vol. 178 (1924), pp. 2051–2052, and Annales de la Faculté des Sciences de l'Université de Toulouse, (3), vol. 17 (1925), pp. 1–22.

§ German *schlicht*.

|| Landau, *Über die Blochsche Konstante und zwei verwandte Weltkonstanten*, Mathematische Zeitschrift, vol. 30 (1929), pp. 608–634; Valiron, *Sur le théorème de M. Bloch*, Rendiconti del Circolo Matematico di Palermo, vol. 54 (1930), pp. 76–82.

shall give a conformal map of the domain of definition, the parameters must be isothermic; that is,

$$(1) \quad E = G, \quad F = 0.$$

With this choice of parameters, by a theorem of Weierstrass, a necessary and sufficient condition that the surface be minimal is that the x_j be harmonic. This being so, these functions are the real parts of analytic functions of a complex variable,

$$(2) \quad x_j = \Re f_j(z), \quad z = u + iv.$$

Equations (1) now are equivalent to

$$(3) \quad \sum_{j=1}^3 f_j'^2 = 0.$$

We shall let these functions (2) represent the mapping in the above theorem. It is no restriction on the generality that we take $f_j(0) = 0$, so that

$$f_j(z) = \sum_{t=1}^{\infty} a_{j,t} z^t.$$

The minimal surface defined by the equation $y_j = \Im f_j(z)$, where $\Im f_j(z)$ designates the imaginary part of $f_j(z)$, is called the adjoint of the minimal surface defined by (2). The surface and its adjoint are applicable to each other, the element of length on each being given by

$$E^{1/2} |dz| \equiv \left[\frac{1}{2} \sum_{j=1}^3 |f_j'|^2 \right]^{1/2} |dz|.$$

LEMMA 1. *Let the circle $|w| < R$ be mapped conformally on a minimal surface, with $\mathcal{E}_0 > 0$, where \mathcal{E}_0 designates the area deformation ratio at the origin. Then it is possible to choose rectangular axes, with origin at the image of $w = 0$, so that in the coordinate functions of the surface,*

$$x_j = \Re g_j(w) = \Re \sum_{t=1}^{\infty} b_{j,t} w^t, \quad w = p + iq,$$

we have

$$(4) \quad |b_{1,1}| = |b_{2,1}| = |b_{3,1}| = (\frac{2}{3} \mathcal{E}_0)^{1/2} = a > 0.$$

Since $\mathcal{E}_0 > 0$, the surface has a definite tangent plane and normal at the image of $w=0$. We choose coordinate axes, with origin at the image of $w=0$, in such a way that the normal at this point has equal components on the three coordinate axes. For these axes, let the coordinate functions of the surface be given by

$$x_j = \mathcal{R}g_j(w) = \mathcal{R} \sum_{t=1}^{\infty} b_{j,t} w^t, \quad |w| < R;$$

then, as in (3),

$$(5) \quad \sum_{j=1}^3 g_j'^2 = 0.$$

Let $X_j(w)$, ($j=1, 2, 3$), denote the direction cosines of the normal to the surface at the point corresponding to w . Because of our choice of axes, we have

$$(6) \quad X_1(0) = X_2(0) = X_3(0) \neq 0.$$

Now

$$g_j' = \frac{\partial x_j}{\partial p} - i \frac{\partial x_j}{\partial q};$$

multiplying this equation through by X_j and summing with respect to j , we get

$$(7) \quad \sum_{j=1}^3 X_j g_j' = 0.$$

Evaluating (7) at $w=0$, and using (6), we obtain

$$(8) \quad \sum_{j=1}^3 g_j'(0) = 0.$$

We have

$$\left(\sum_{j=1}^3 g_j' \right)^2 = \sum_{j=1}^3 g_j'^2 + 2 \sum_{j < k} g_j' g_k',$$

so that, by (5) and (8),

$$(9) \quad \sum_{j \leq k} g_j'(0) g_k'(0) = 0.$$

Let $\alpha^3 + m_1\alpha^2 + m_2\alpha + m_3 = 0$ be the equation whose roots are $g'_1(0)$, $g'_2(0)$, $g'_3(0)$. Then, by (8) and (9), $m_1 = m_2 = 0$, so that $\alpha^3 + m_3 = 0$. That is, $g'_1(0)$, $g'_2(0)$, $g'_3(0)$ lie on a circle with center at the origin, or, what is the same thing, $|b_{1,1}| = |b_{2,1}| = |b_{3,1}|$. From

$$\mathcal{E} = \frac{1}{2} \sum_{j=1}^3 |g'_j|^2,$$

we therefore obtain (4).

LEMMA 2. *Let the functions*

$$(10) \quad g_j(w) = \sum_{t=1}^{\infty} b_{j,t} w^t, \quad (j = 1, 2, 3),$$

be analytic for $|w| < R$, with

$$\sum_{j=1}^3 g_j'^2 = 0, \quad \frac{1}{2} \sum_{j=1}^3 |b_{j,1}|^2 = \mathcal{E}_0 > 0,$$

and

$$(11) \quad |g_j(w)| \leq 2^{1/2} M.$$

Then on the minimal surface defined by $x_j = \Re g_j(w)$ there is an open geodesic circle whose center is the image of the origin, whose radius is at least $R^2 \mathcal{E}_0 / (6 \cdot 3^{1/2} \cdot M)$, and which contains no singular points. Similarly, there is a geodesic circle of the same description on the adjoint minimal surface.

The condition (11) necessitates that both the surface and its adjoint be contained in a sphere with center at the origin and radius $6^{1/2} M$; conversely, (11) surely is satisfied if both surfaces are contained in a sphere with center at the origin and radius M .

By simply rotating the coordinate axes, we can normalize the functions (10) in accordance with Lemma 1, without altering the conditions of Lemma 2. We consider this done and still use the expressions (10), so that (4) holds for the functions (10).

Landau has shown* that $g_j(w)$ gives a uniplanar map of

$$(12) \quad |w| \leq \frac{aR}{4 \cdot 2^{1/2} \cdot M} = b,$$

and that this map contains the uniplanar open circle

* Loc. cit., pp. 616-617.

$$|g_j| < \frac{a^2 R^2}{6 \cdot 2^{1/2} \cdot M},$$

the center of this circle being the image of the origin.

The length of a curve on the minimal surface (10) or on its adjoint is given by the equation

$$L = \int \left[\frac{1}{2} \sum_{j=1}^3 |g'_j|^2 \right]^{1/2} |dw|.$$

By Minkowski's inequality,*

$$L^2 \geq \frac{1}{2} \sum_{j=1}^3 \left[\int |g'_j| \cdot |dw| \right]^2,$$

the path of integration being the same throughout. We have just seen that, for all paths of integration,

$$\text{minimum} \int_0^b |g'_j| \cdot |dw| \geq \frac{a^2 R^2}{6 \cdot 2^{1/2} \cdot M},$$

so that, a fortiori,

$$\text{minimum} \int_0^b \left[\frac{1}{2} \sum_{j=1}^3 |g'_j|^2 \right]^{1/2} |dw| \geq \frac{3^{1/2} \cdot a^2 R^2}{2 \cdot 6 \cdot M} = \frac{R^2 \mathcal{E}_0}{6 \cdot 3^{1/2} \cdot M}.$$

This demonstrates the existence of the prescribed geodesic circle. Singular points on the minimal surface occur only where $\mathcal{E} = 0$; that is, where simultaneously $g'_1 = g'_2 = g'_3 = 0$. But $g_j(w)$ gives a uniplanar map of (12) and therefore $g'_j \neq 0$ in this region. It follows that $\mathcal{E} \neq 0$ in (12).

LEMMA 3.† Let ξ be a constant, $|\xi| < 1$, and let

$$z = \frac{\xi + w}{1 + \bar{\xi}w}.$$

Then, for $|w| < 1$,

$$\left| \frac{dz}{dw} \right| = \frac{1 - |z|^2}{1 - |w|^2}.$$

* See, for instance, Pólya und Szegő, *Aufgaben und Lehrsätze*, vol. I, 1925, p. 56, §91.

† For the elementary proof, see for instance Landau, loc. cit., p. 617.

We come now to the proof of the theorem. We can assume that $E_0 = 1$, because if the theorem holds in this case it holds a fortiori for $E_0 > 1$.

Let

$$\begin{aligned} N &= \text{maximum}_{|z| \leq 1} (1 - |z|^2) \left[\frac{1}{2} \sum_{j=1}^3 |f'_j|^2 \right]^{1/2} \\ &= \text{maximum}_{|z| \leq 1} (1 - |z|^2) E^{1/2}. \end{aligned}$$

Then $N \geq 1$, and N is attained for some $z = \xi$, with $|\xi| < 1$. The unit circle is mapped conformally on itself by

$$z = \frac{\xi + w}{1 + \bar{\xi}w},$$

so that the functions

$$g_i(w) = \frac{f_i(z)}{N}$$

are analytic for $|w| \leq 1$. And the real parts of these functions map $|w| \leq 1$ conformally on a minimal surface. By Lemma 3, for $|w| < 1$, we have

$$\begin{aligned} (1 - |w|^2) |g'_j(w)| &= (1 - |w|^2) \frac{1}{N} |f'_j(z)| \cdot \left| \frac{dz}{dw} \right| \\ &= \frac{1}{N} (1 - |z|^2) |f'_j(z)|, \end{aligned}$$

whence

$$\begin{aligned} (13) \quad (1 - |w|^2) \left[\frac{1}{2} \sum_{j=1}^3 |g'_j(w)|^2 \right]^{1/2} \\ = \frac{1}{N} (1 - |z|^2) \left[\frac{1}{2} \sum_{j=1}^3 |f'_j(z)|^2 \right]^{1/2} \leq 1, \end{aligned}$$

so that

$$\left[\frac{1}{2} \sum_{j=1}^3 |g'_j(0)|^2 \right]^{1/2} = \mathcal{E}_0 = \frac{1}{N} (1 - |\xi|^2) \left[\frac{1}{2} \sum_{j=1}^3 |f'_j(\xi)|^2 \right]^{1/2} = 1.$$

From (13) we obtain also, for $|w| \leq 1/2$,

$$\left[\frac{1}{2} \sum_{j=1}^3 |g'_j(w)|^2 \right]^{1/2} \leq \frac{4}{3},$$

so that, integrating along any radius, we have

$$L = \int_0^{1/2} \left[\frac{1}{2} \sum_{j=1}^3 |g'_j(w)|^2 \right]^{1/2} dr \leq \frac{2}{3}.$$

Therefore the minimal surfaces given by $x_j = \Re g_j(w)$ and $y_j = \Im g_j(w)$ for $|w| \leq 1/2$ both are contained in a sphere with center at the origin and radius $2/3$.

Consequently, we can apply Lemma 2 to the functions $g_j(w)$, with $R=1/2$, $M=2/3$, $\mathcal{E}_0=1$. We see, namely, that on the map of $|w| < 1/2$ given by

$$x_j = \Re g_j(w), \quad (j = 1, 2, 3),$$

there is an open geodesic circle of radius at least $1/(16 \cdot 3^{1/2})$ and comprising no singular points; and therefore that on the map of $|z| < 1$ given by (2) there is an open geodesic circle of radius at least

$$\frac{N}{16 \cdot 3^{1/2}} \geq \frac{1}{16 \cdot 3^{1/2}}$$

and comprising no singular points.

THE OHIO STATE UNIVERSITY