A NEIGHBORHOOD TREATMENT OF GENERAL TOPOLOGICAL SPACES*

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In this paper we deal with all the subsets of a space R of elements called points. Each point p of R may have associated with it certain subsets of R called *neighborhoods of* p so that it is determined by some rule whether or not any particular set is a neighborhood of any particular point. In the most general case no assumptions are made such as that every point has at least one neighborhood associated with it, or that the point p is an element of the neighborhood associated with it.

The purpose of this paper will be to consider (1) the various ways of defining limit point in terms of neighborhoods; (2) what properties must be assumed concerning the neighborhoods in order that limit point have certain well known properties. We shall start with the following new definition of limit point which, although somewhat peculiar in character, is found to be most convenient for the case of the general topological space.†

DEFINITION A. A point p is said to be a *limit point* of a set E if every neighborhood of p that contains $C(E)$‡ contains at least one point of E.

DEFINITION B. A point p is said to be *interior* to the set E if it is not a limit point of $C(E)$.

THEOREM 1. *The set* N *is a neighborhood of the point* p *if and only if* p *is interior to* N.

PROOF. (1) Suppose p were not interior to N; then p would be a limit point of $C(N)$ and hence every neighborhood of p that contains the complement of $C(N)$ must contain a point of $C(N)$. Now N is a neighborhood of p by hypothesis, but it contains no point of $C(N)$, which gives us a contradiction. (2) If p is not a limit point of $C(N)$, then there is a neighborhood of p which contains N but no point of $C(N)$, hence N is the neighborhood.

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† M. Fréchet, *Les Espaces Abstraites*, p. 166; E. W. Chittenden, Transactions of this Society, vol. 31 (1929), pp. 290–321.

‡ The symbol $C(E)$ means the complement of the set E. The whole definition implies that p is a limit point of E if $C(E)$ is not a neighborhood of p.

From this theorem it can be seen that with Definition A the general topological space can be considered as a neighborhood space. For if we take for the neighborhoods of each point p all sets to which p is interior, then by Theorem 1, Definition A will define limit point correctly.

DEFINITION C. A space is said to be *monotonic** if it has the property that for every point p and set A, if p is a limit point of A it is a limit point of every set containing A. This can be written in symbols:†

If $A < B$, then $A' < B'$, or $(A+B)' > A'+B'$.

THEOREM 2. *The space R is monotonic if and only if it has the property that if the set N is a neighborhood of the point p, so is every set containing N.*

The proof of Theorem 2 follows easily from Definition C and will be omitted.

DEFINITION D. A space is said to have the *partition property*‡ if whenever p is a limit point of $(A+B)$, it is a limit point of either A or B. In symbols $(A+B)' < A'+B'$.

THEOREM 3. *A necessary and sufficient condition that a space R have the partition property is that if M and N are neighborhoods of the point p, so is their logical product $M \cdot N$.*

PROOF. First, suppose p were not a limit point of either A or B. Then $C(A)$ and $C(B)$ would be neighborhoods of p and so would their logical product $C(A) \cdot C(B) = C(A+B)$; then $C(A+B)$ would be a neighborhood of p and consequently p could not be a limit point of $A+B$, contrary to hypothesis. Secondly, suppose $M \cdot N$ were not a neighborhood; then p is a limit point of $C(M \cdot N)$ which is equal to $C(M)+C(N)$, and hence p is a limit point either of $C(M)$ or $C(N)$, so that neither M nor N could be a neighborhood of p, contrary to hypothesis.

DEFINITION E. A set is *closed* if it contains all of its limit points.

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* Dorothy McCoy, Tôhoku Mathematical Journal, vol. 33 (1930), pp. 88–116.

† The symbol $<$ in $A < B$ means "A is contained in B". The symbol A' means the set of limit points of the set A. The symbol $>$ in $(A+B)' > A'+B'$ means that $(A+B)'$ includes $A'+B'$.

‡ See Dorothy McCoy, loc. cit.

DEFINITION F. A set is said to be *open* if every point of it is interior to it. Thus an open set is the complement of a closed set, and conversely.

DEFINITION G. The set of all limit points of the set A is called the *derived set*, A'.

DEFINITION H. If all the derived sets in the space R are closed, that is, if $(A')' < A'$, the space R is said to be *accessible*.

THEOREM 4. *A space R is accessible if and only if it is true that whenever N is a neighborhood of a point p, then the set of all points of which N is a neighborhood is also a neighborhood of the point p.*

PROOF. First, let I be the set of all points of which N is a neighborhood. By Theorem 1, I consists of all points that are not limit points of $C(N)$; hence $C(I) = (C(N))'$ which is closed by hypothesis. Hence $C(I)$ is closed and therefore I is open, and, consequently, I is a neighborhood of p. Secondly, suppose p is a limit point of A' but not of A; then $C(A)$ is a neighborhood of p, and so I, which is the set of all points of which $C(A)$ is a neighborhood, is a neighborhood of p by hypothesis, but $I = C(A')$ and hence p is not a limit point of A', contrary to hypothesis.

COROLLARY. *The space R is accessible if and only if the set of all points interior to any set is an open set.*

DEFINITION I. A space R is said to have the *non-singular** property if whenever p is a limit point of the set A it is a limit point of $A - p$.

THEOREM 5. *The space R is non-singular if and only if whenever p is a limit point of the set A, every neighborhood of p which contains $C(A)$ contains a point of A other than p.*

PROOF. First, suppose there exists a neighborhood of p which includes $C(A)$ but contains no point x of A other than p. The complement of this neighborhood is equal to A or $A - p$; p is a limit point of the complement of this neighborhood N and therefore N could not be a neighborhood of p, contrary to assumption. Secondly, suppose the space R were non-singular; then $C(A - p)$ which is equal to $C(A) + p$ would be a neighborhood of p and therefore contain a point of A other than p, but this is impossible.

* Dorothy McCoy, loc. cit.

THEOREM 6. *The space R is monotonic if and only if whenever p is a limit point of the set A, every neighborhood of the point p contains a point of A.*

This follows easily and the proof will be omitted.

DEFINITION J. A space R is said to have the *infinite* property if whenever p is a limit point of the set A it is a limit point of the set $A-x$, x being a set containing a single point.

THEOREM 7. *The space R has the infinite property if and only if whenever p is a limit point of the set A, every neighborhood of p which contains the complement of A contains an infinite number of points of A.*

PROOF. First, suppose there exists a neighborhood of p, N_p, which contains $C(A)$ and has only a finite number of points of the set A. Call this finite set F. Now p is a limit point of $A-F$ by hypothesis and $N_p=C(A-F)$ which is equal to $C(A)+F$, and therefore N_p must contain a point of $A-F$ by the definition of limit point, but this is impossible since N_p contains only $C(A)$ and the finite set F. Secondly, suppose p is a limit point of $A+F$ (the set F being finite) but not of A. Then $C(A)$ is a neighborhood of p and contains $C(A+F)$ and hence by hypothesis must have an infinite set of points of $A+F$, but these must be in F, which is finite, and so we have a contradiction.

DEFINITION K. Two systems of neighborhoods $\{M_p\}$ and $\{N_p\}$ of a point p are said to be *equivalent* if every M_p contains an N_p and every N_p contains an M_p.

THEOREM 8. *If in a monotonic space R, a system of neighborhoods $\{M_p\}$ of the point p is equivalent to the set of all neighborhoods $\{N_p\}$ of p, then p is a limit point of the set S if and only if every M_p contains a point of the set S.*

This follows easily from Theorem 6 and the proof will be omitted.

CONCLUSION. It is possible to define limit point in terms of neighborhood in the following four ways, each of which is, in a sense, more special than the preceding.

1. A point p is a limit point of the set S if every neighborhood of p which contains the complement of S also contains a point of the set S.

2. A point p is a limit point of the set S if every neighborhood of p contains a point of the set S.

3. A point p is a limit point of the set S if every neighborhood of p contains a point of the set S other than p.

4. A point p is a limit point of the set S if every neighborhood of p contains an infinite number of points of the set S.

Definition 1 is the one we have used throughout. Although unusual in its form, it has the advantage that it implies no properties of limit point and hence can be used in the most general topological spaces. In particular it does not imply the monotonic property, which distinguishes it from all previous neighborhood definitions of limit point. Hence it can be used in connection with the notion of sequential limit point, which does not have the monotonic property, or with the notion of boundary point. Other non-monotonic relations between point and set will suggest themselves, which can now be treated in terms of neighborhood, using Definition 1.

Definition 2 is the simplest of these definitions. It obviously implies the monotonic property. Theorem 6 shows that every monotonic space can be treated as a neighborhood space using this definition of limit point. Although Definition 2 is very simple and leads to a very neat treatment of the most general monotonic space, it does not seem to have been considered in the literature. This definition acquires added importance when it is remembered that Chittenden (loc. cit.) has shown that in the most general topological space one can redefine limit point so as to make the space monotonic.

Definition 3 is one of the usual ways of defining limit point in terms of neighborhood and is used by Fréchet* for his very general V-spaces. If we use this definition the space is necessarily monotonic and non-singular. In connection with Theorem 1, Theorems 5 and 6 show that in any monotonic and non-singular space we can define limit point in terms of neighborhood using Definition 3.

Similarly Definition 4 implies the monotonic and infinite properties, and Theorems 6 and 7 show that this definition, with neighborhoods defined as suggested by Theorem 1, can be used in the most general space having these two properties.

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* *Les Espaces Abstraites*, p. 172.