NOTE ON SETS OF POSITIVE MEASURE*

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A recurring question concerning (L-measurable) sets of positive measure is what properties they have in common with the linear interval. The following theorem is concerned with such a property, stated for sets of n-dimensional positive measure lying in euclidean n space.

THEOREM. *Let $A_1, A_2, \cdots, A_p$ be p sets of positive measure lying in euclidean n space. Then there exist p n-dimensional spheres $S_1, S_2, \cdots, S_p$ such that for every set of p points s_ν, $(\nu=1,2,\cdots,p)$, belonging respectively to these spheres, there exists a set of p points a_ν, $(\nu=1, 2, \cdots, p)$, lying respectively in A_1, $A_2, \cdots, A_p$, such that the sets $\{a_\nu\}$ and $\{s_\nu\}$ are congruent. Moreover, there exists a set of p congruent spheres S_ν satisfying the condition just stated and a positive number δ such that for every selected $\{s_\nu\}$, with s_ν belonging to S_ν, the associated $\{a_\nu\}$ may be so chosen that a_1 ranges over a set of measure $>\delta$.*

PROOF. Since A_ν is of positive measure, there is a sphere S_ν' in which the relative measure of A_ν is greater than $1-\epsilon$, where ϵ is a given positive number less than 1; that is, $m(A_\nu, S_\nu')/m(S_\nu') > 1-\epsilon$, $m(A)$ standing for the measure of A. We may suppose, and we do so for simplicity of statement, that all the S_ν', $(\nu=1, \cdots, p)$, are equal, and we denote their common measure by μ, and their respective centers by c_ν. Let ρ be a positive number such that if a sphere of measure μ is translated a distance $<\rho$, the part belonging to the sphere in both positions is of measure $>(1-\epsilon)\mu$. Denote by v_ν, $(\nu=1, \cdots, p-1)$, the vector represented by the segment $c_\nu c_{\nu+1}$; and let w_ν, $(\nu=1, \cdots, p)$, be a given set of n-dimensional vectors, each of length $<\rho$. If a set A (or point a) is given a displacement represented by the vector $\pm v$, we denote the set (or point) in its new position by $A \pm (v)$ (or $a \pm (v)$). Writing $A_\nu S_\nu' = T^{(\nu)}$ and $T' = T_1'$, we set

$$T_1' + (v_1 - w_1 + w_2) = T_2' ; \; T_2' T'' = T_1'' ;$$

* Presented to the Society, April 14, 1923.

$$T_1'' + (v_2 - w_2 + w_3) = T_2' \ ; \ T_2''T''' = T_1''' \ ;$$

. . . .

$$T_1^{(p-1)} + (v_{p-1} - w_{p-1} + w_p) = T_2^{(p-1)}; \ T_2^{(p-1)}T^{(p)} = T_1^{(p)}.$$

For the measures of the $T_\lambda^{(\nu)}$'s, we have the following inequalities: $m(T_1') > (1-\epsilon)\mu$; $m(T_1'') > (1-4\epsilon)\mu$, since $m(T'') > (1-\epsilon)\mu$ and the lengths of w_1 and w_2 are less than ρ; $m(T_1''') > (1-7\epsilon)\mu$, and so on. We may thus conclude that $m(T_1^{(p)}) > [1-(3p-2)\epsilon]\mu$, which is positive if ϵ is taken small enough. We now define the spheres $S_1, \cdots, S_p$ of our theorem as of radii all less than ρ, and such that their respective centers γ_ν satisfy the relations: vector $\gamma_\nu\gamma_{\nu+1} = v_\nu$, $(\nu = 1, \cdots, p-1)$. If now s_ν, $(\nu = 1, \cdots, p)$, is a point chosen from S_ν, we let vector $\gamma_\nu s_\nu = w_\nu$. Let a_p be a point of $T_1^{(p)}$, which, as we have seen, is not empty if ϵ is sufficiently small. We then define $a_{p-1}, a_{p-2}, \cdots, a_1$ by the relation

$$a_\nu = a_{\nu-1} + (v_{\nu-1} - w_{\nu-1} + w_\nu), \qquad (\nu = 2, 3, \cdots, p).$$

Since a_p belongs to $T_1^{(p)}$, it belongs to A_p and also to $T_2^{(p-1)}$; hence a_{p-1} belongs to $T_1^{(p-1)}$ and therefore to A_{p-1} and $T_2^{(p-2)}$; hence a_{p-2} belongs to $T_1^{(p-2)}$, and so on. We conclude that a_ν, $(\nu = 1, \cdots, p)$, belongs to A_ν. Since s_ν satisfies the equation $s_\nu = s_{\nu-1} + (v_{\nu-1} - w_{\nu-1} + w_\nu)$, $(\nu = 2, \cdots, p)$, we see that the sets $\{a_\nu\}$ and $\{s_\nu\}$ are congruent. Furthermore, since a_p is an arbitrary point of $T_1^{(p)}$, whose measure is arbitrarily near μ, we can satisfy the last condition of our theorem by taking, for example, $\delta = \mu/2$, if ϵ is small enough.

If, in particular, the p given sets A_ν are identical, we may take the spheres S_ν' as identical, thus reducing the vectors v_ν to zero. The spheres S_ν may therefore be taken as identical, and we have the following corollary.

COROLLARY. *If A is an n-dimensional set of positive measure, there exists an n-dimensional sphere S, such that for every finite subset of S there is a congruent subset of A.*

If A is a one-dimensional set, we obtain the theorem of Steinhaus,* that *the set of distances between pairs of points of a*

* *Sur les distances des points*, Fundamenta Mathematicae, vol. 1 (1920), p. 99. A simpler proof of this theorem, close in idea to our own, was published by Ruziewicz (after the present paper was read), Fundamenta Mathematicae, vol. 7 (1925), p. 141.

(linear) set of positive measure contains an interval with 0 *as left end point.*

We have proved that if S_1 is a set of positive measure and S_2 a finite set, there is a subset of S_1 similar* to S_2. To what extent can the condition of finiteness of S_2 be modified if the theorem is to remain valid? Since every set of positive measure contains a perfect subset of positive measure, it follows that if every set of positive measure contains a set similar to S_2, it contains a set similar to S_2+S_2', where S_2' is the derivative of S_2. It thus suffices to restrict S_2 to being closed. Or we may restrict S_2 to being denumerable, since every set contains a denumerable subset which is dense in it. Not all sets of positive measure can contain a set similar to S_2 if S_2 is not non-dense. Since we naturally restrict S_2 to being bounded, we are led to ask: What bounded, non-dense sets S_2 are such that every set of positive measure contains a set similar to S_2? That this property is not shared by every bounded, non-dense S_2, and therefore not by every bounded, non-dense, denumerable set, is shown by the following fact.

THEOREM. *If S_1 is a given bounded, non-dense, perfect set, there exists a perfect set S_2 of zero measure such that no subset of S_1 is similar to S_2.*

While this theorem is meant to refer to n-dimensional sets, we assume in the proof that S_1 and S_2 are linear sets, there being no essential difference in the argument for n-dimensional sets. We suppose, as we may, that the given set S_1 lies in the interval $(0, 1)=I$. Let $C(S_1)$ be the complement of S_1 in I; Δ a variable subinterval of I; $\mu_1(\Delta)$ the ratio of the maximum length of a connected portion of $C(S_1)$ in Δ to the length of Δ; and $\sigma_1(h)$, for h a given positive number, the greatest lower bound of $\mu_1(\Delta)$ for all subintervals Δ of I of length h. Then $\sigma_1(h)$ is a positive, continuous function of h. The perfect set S_2 will be defined as the complement in I of the set of intervals Δ_{ni}, which are defined as follows: Insert in I a set of equally spaced intervals Δ_{1i}, $(i=1, 2, \cdots, m_1)$, of equal length l_1, such that $m_1l_1=1/2$, the equality of spacing being understood in the sense that the

* We are using "similar" in the ordinary euclidean sense of the existence of a biunique correspondence with invariant ratio of distances.

space between any two adjacent Δ_{1i} shall be equal to the spaces between 0, 1 and the first, last Δ_{1i}, respectively; moreover, m_1 is to be so large that $\sigma_2(\lambda_1) < \sigma_1(\lambda_1)$, where $\lambda_1 = 1$, and σ_2 has the same meaning for the set S_2, now being defined, as σ_1 for S_1. Similarly insert in each of the intervals Δ_{1i}', $(i = 1, 2, \cdots, m_1 + 1)$, of length l_1', that are complementary to the Δ_{1i} the same number of equally-spaced intervals Δ_{2i}, $(i = 1, 2, \cdots, m_2)$, of equal length l_2, where m_2 signifies the total number of the Δ_{2i} in all the Δ_{1i}'; moreover, the Δ_{2i} are to be such that $m_2 l_2 = 1/4$, and m_2 so large that $\sigma_2(t\lambda_2) < \sigma_1(\lambda_2)$ for $1 \leqq t \leqq 2$, where $2\lambda_2 = l_1'$. In general, let $\{\Delta_{n-1,i}'\}$ be the set of intervals of length l_{n-1}', complementary to the set of all $\Delta_{\nu i}$, $\nu \leqq n-1$. Insert in each $\Delta_{n-1,i}'$ the same number of equally spaced intervals Δ_{ni} of equal length l_n, such that $m_n l_n = 1/2^n$, m_n being the total number of intervals Δ_{ni} and m_n so large that $\sigma_2(t\lambda_n) < \sigma_1(\lambda_n)$ for $1 \leqq t \leqq n$, where $n\lambda_n = l_{n-1}'$. Suppose now that S_3 is any set whatsoever lying in I and similar to S_2; then S_3 cannot lie in S_1. For let k be the ratio of corresponding lengths in S_2 and S_3, and n an integer greater than k. If ϵ is a given positive number, we can find an interval Δ of length λ_n lying between two points of S_3, and such that $|\mu_3(\Delta) - \sigma_2(k\lambda_n)| < \epsilon$, μ_3 having the same meaning for S_3 as μ_1 for S_1. Hence, on account of the inequality $\sigma_2(k\lambda_n) < \sigma_1(\lambda_n)$, we conclude that $\mu_3(\Delta) < \sigma_1(\lambda_n) \leqq \mu_1(\Delta)$, if ϵ is small enough. That is to say, the maximum length of a connected portion of $C(S_3)\Delta$ is less than such maximum length for $C(S_1)\Delta$, and therefore S_3 cannot lie in S_1.

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