LANE ON PROJECTIVE DIFFERENTIAL GEOMETRY

Projective Differential Geometry of Curves and Surfaces. By Ernest Preston Lane. Chicago, The University of Chicago Press, 1932. xi+321 pp.

The author's aim in writing this treatise is clearly set forth in the preface: "Projective differential geometry is largely a product of the first three decades of the twentieth century. The theory has been developed in five or more different languages, by three or four well recognized methods, in various notations, and partly in journals not readily accessible to all." It is Lane's purpose "to organize an exposition of these researches." He "desires to coordinate the results achieved on both sides of the Atlantic so as to make the work of the European geometers more readily accessible to American students, and so as to make better known to others the accomplishments of the American school."

Differential geometry is defined as "the theory of the properties of a configuration in the neighborhood of a general one of its elements." Such properties were known long before the name of projective differential geometry was invented. In fact algebraic geometry, which is as old as differential geometry, has been and is using projective differential notions very frequently.

An excellent confirmation of this statement may be found in the first chapter of the book under review, in connection with the theory of space cubics and the nullsystems attached to them, which also serves as a classical example in algebraic geometry. These ideas are extended to space curves in general and we find among others the interesting theorem: *The nullsystem of the osculating linear complex at a point P of a curve C is the same as the nullsystem of the osculating twisted cubic of C at P.*

Here is a typical proposition of projective differential geometry: *In a space S_n the $\infty^{n(n+2)}$ integral curves of a given differential equation*

$$x^{(n+1)} + (n + 1)p_1x^{(n)} + \cdots + p_{n+1}x = 0$$

are all projectively equivalent. A differential equation (of this sort) defines a curve in the space S_n except for a projective transformation, and a geometric theory based on the equation must be a projective theory.

The study of projective *invariants* and *covariants* in differential geometry is, of course, of great importance. A projective invariant of an integral curve is defined as a function of the coefficients of the preceding differential equation, and of their derivatives, which is invariant under the total transformation $x=\lambda(t)\zeta$ (λ scalar $\neq 0$). Every absolutely invariant equation connecting these invariants is independent of the analytic representation of the curve and expresses a projective geometric property of the curve. A *covariant* defines a curve whose points are in (1, 1) correspondence with the points of the original curve and obtained from the latter by a projective geometric construction.

The systematic development of these theories goes back to the remarkable geometrician, the lamented Wilczynski, who in his memorable projective differential geometry published by Teubner in 1906, mentions the admirable theory developed by Halphen. In presenting these theories Wilczynski however followed his own methods, "both for the sake of uniformity and simplicity."

Chapter 2 deals with ruled surfaces, including developable surfaces. Some of the propositions have been known for a long time, for example the theorem by Chasles: *The cross ratio of four points on a generator of a non-developable ruled surface is equal to the cross ratio of the four tangent planes to the surface at these points.* We find a treatment of Wilczynski's system for differential equations of a ruled surface in S_3; of osculating curves and the osculating quadric, flecnode curves and the flecnode transformation, the osculating linear complex of a ruled surface in S_3, and so forth.

Surfaces in ordinary space form the contents of the next chapter. A characterization of the methods of Wilczynski and Fubini for the development of extensive theories of projective differential geometry will be found very helpful in understanding and appreciating the following sections, which deal with topics like these: The differential equations of a surface referred to its asymptotic net, the local coordinate system, power series expansions, Darboux quadrics, reciprocal congruences, the canonical pencil, conjugate nets, Cěch's transformation, Demoulin's tetrahedron.

The theory of *conjugate nets*, as developed in Chapter 4, occupies an important place in projective differential geometry, as attested by the large number of mathematicians contributing to this theory, including the Americans Wilczyski, Green, Eisenhart, to which Lane ought to add himself.

It is perhaps well to state what is meant by this concept: *A net of curves on a surface in a linear space of n dimensions is a conjugate net in case the tangents of the curves of one family of the net constructed at the points of each fixed curve of the other family form a developable surface.*

Space hardly permits a detailed discussion of the rich and interesting contents of this chapter, nor of those of the next chapter, dealing with transformation of surfaces. In Chapter 6 the author aims to connect the projective differential geometry of surfaces, as developed in the earlier chapters of the book, with the metric and affine differential geometries of surfaces of ordinary space and Chapter 7 contains some further developments for surfaces in hyperspace under the title of surfaces and varieties. The last chapter is concerned with miscellaneous topics, beginning with some historical remarks.

The book is clearly written and the typography is good. For student and investigator alike in this field it will be found indispensable.

As to the general importance of the old projective differential geometry, one should keep in mind that it forms a relatively small branch of the vast field of differential geometry, especially in view of the tremendous modern development due to the impetus originally given it by the advent of the general relativity theory. The relation is somewhat similar to that of projective geometry with respect to the vastly more extensive field of algebraic geometry based upon the birational group and given strong support by modern topology.

The new "projective differential geometry" as will be pointed out by D. van Dantzig in a paper to be published in the Mathematische Annalen, gives a theory of projective connection of n-dimensional space, containing the theories of Cartan, Weyl, Veblen, and Schouten as special cases and which may also be used for the establishment of projective Riemannian differential geometry and also projective Cayley-Klein differential geometry.

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