ERROR EXPRESSIONS FOR CERTAIN CONTINUED FRACTIONS*

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Every discussion of the subject shows that the error committed in taking the nth convergent p_n/q_n as the true value of the simple continued fraction

$$(1) \quad a_1 + \cfrac{1}{a_2 +} \cfrac{1}{a_3 +} \cdots + \cfrac{1}{a_n +} \cdots , \qquad (a_i \text{ positive integers}),$$

is less than $1/(q_n q_{n+1})$, and greater than $a_{n+2}/(q_n q_{n+2})$ in absolute value. Further, p_n/q_n is smaller than the true value if n is odd, and larger if n is even, provided the continued fraction does not terminate.

The purpose of the present paper is to supply alternate error limits for the important case when the continued fraction is the expansion of the square root of an integer N. These limits have the advantage that they do not require the computation of any convergent beyond the nth. Further, when applied as corrections to the nth convergent, they practically always lead to a much closer approximation than do the general error limits. Our results may be summarized as follows:

Given $N^{1/2}$ expressed in the form (1), with nth convergent p_n/q_n, and with k defined† as $\left| p_n^2 - Nq_n^2 \right|$. If n is even, the error committed in taking p_n/q_n as the value of $N^{1/2}$ is greater than $k/(2p_n q_n)$, and less than $k/(2q_n^2 N)^{1/2}$, in absolute value. If n is odd, this statement is true with "greater than" and "less

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† We might affix a subscript to k, but it does not seem necessary. If p_n and q_n are large the arithmetic required to compute k might be considerable if it were necessary to multiply out $p_n^2 - Nq_n^2$. This is not, however, usually necessary. For, as is well known, the values that k can take on come in cycles, and the complete cycle can be formed easily from small values of n in many cases. It is also known that k is less than $2N^{1/2}$, which will render the complete squaring of p_n and q_n unnecessary even when the complete cycle of values has not been formed. Finally, if $n = mc$, where m is a positive integer and c is the number of partial quotients in the period of the continued fraction expansion of $N^{1/2}$, then k is 1. This case will receive special mention at the end of the present paper.

than" interchanged. These limits are ordinarily more satisfactory than the limits $1/(q_n q_{n+1})$ and $a_{n+2}/(q_n q_{n+2})$.

To obtain the desired error expressions we have merely to write

$$E \equiv \mid p_n/q_n - N^{1/2} \mid = k/(q_n(p_n + N^{1/2}q_n)).$$

Now if n is even, and we replace $N^{1/2}$ by p_n/q_n, which is larger than it, we have at once $E > k/(2p_n q_n)$. And if we, instead, replace p_n by $N^{1/2}q_n$, which is smaller than it, we find $E < k/(2q_n^2 N^{1/2})$. To render this upper limit convenient for computation we may now replace $N^{1/2}$ by any approximation which is too small, say p_{n-1}/q_{n-1} or the better approximation Nq_n/p_n. If the $(n+1)$st convergent can be found easily it will be still better. The reader will see at once the modifications which must be made in these statements when n is odd.

It seems desirable to compare these error expressions with the general limits $1/(q_n q_{n+1})$ and $a_{n+2}/(q_n q_{n+2})$. First, the upper limit just determined is always less than $1/(q_n q_{n+1})$, when n is even. To see this note that the inequality

$$(2) \qquad \frac{k}{q_n(p_n + N^{1/2}q_n)} < \frac{1}{q_n q_{n+1}}$$

implies that

$$(3) \qquad kq_{n+1} < p_n + N^{1/2}q_n < 2p_n - \frac{a_{n+2}}{q_{n+2}}.$$

Since kq_{n+1} is an integer and $a_{n+2} < q_{n+2}$, we then have $kq_{n+1} \leqq 2p_n - 1$. Now $k/(2q_n^2 N^{1/2})$ is less than $1/(q_n q_{n+1})$ provided $kq_{n+1} < 2N^{1/2}q_n$; and this inequality will be established as soon as we show that $2p_n - 1 < 2N^{1/2}q_n$. This last is easily seen to be reducible to $E < 1/(2q_n)$, which is satisfied since E is certainly less than $1/(q_n q_{n+1})$, and $q_{n+1} \geqq 2$ when n is even.

We have mentioned that $N^{1/2}$ may be replaced by suitable approximations to give a more convenient, though poorer, upper limit. Suppose we use p_{n-1}/q_{n-1}. A sufficient condition that the new upper limit still be less than $1/(q_n q_{n+1})$ is that $2p_n - 1$ be less than $2q_n p_{n-1}/q_{n-1}$. Using the relation $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$ this reduces to $2 < q_{n-1}$. Hence the modified upper limit is less than $1/(q_n q_{n+1})$ unless q_{n-1} is 1 or 2; it may even be less in these trivial cases. The same statement holds, a fortiori, when

the modified upper limit is formed with a better approximation than p_{n-1}/q_{n-1}.

The lower limit $k/(2p_nq_n)$ for n even is greater than or equal to $a_{n+2}/(q_nq_{n+2})$ provided $a_{n+2} \leqq q_{n+1}$. This sufficient condition will certainly be satisfied unless n is small, for it is known that a_{n+2} is less than $2N^{1/2}$. To establish this relation between the lower limits note that the inequality

$$(4) \qquad \frac{k}{q_n(p_n + N^{1/2}q_n)} > \frac{a_{n+2}}{q_nq_{n+2}}$$

implies that

$$(5) \qquad kq_{n+2} > a_{n+2}(2p_n - Eq_n).$$

Now $k/(2p_nq_n) \geqq a_{n+2}/(q_nq_{n+2})$ provided $kq_{n+2} \geqq 2p_na_{n+2}$, and it follows from (5) that this condition will be satisfied whenever $a_{n+2}Eq_n \leqq 1$. But since $E < 1/(q_nq_{n+1})$, the sufficient condition as stated follows.

It is not easy to set up a simple example in which the new lower limit is poorer than the old. One can be found in connection with

$$(6) \quad 31^{1/2} = 5 + \frac{1}{1} + \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{3} + \frac{1}{1} + \frac{1}{1} + \frac{1}{10} + \cdots$$

If the second convergent, 6, be taken as the value of $31^{1/2}$, the error is greater than $5/12$ by the expression $k/(2p_nq_n)$, and greater than $3/7$ by the expression $a_{n+2}/(q_nq_{n+2})$. In this case, of course, we have $a_{n+2} > q_{n+1}$. If the fourth convergent be used instead of the second the two lower limits give the same value, but our new upper limit is better for the sixth convergent.

We turn to the consideration of the case where n is odd. An odd convergent is less in value than the continued fraction, so we have $p_n/q_n = N^{1/2} - E$, and in place of (3) we find

$$(7) \qquad kq_{n+1} < 2p_n + Eq_n < 2p_n + \frac{1}{q_{n+1}}.$$

It then follows that $kq_{n+1} \leqq 2p_n$. And this is exactly the condition necessary to insure that the new upper limit $k/(2p_nq_n)$ is not greater than $1/(q_nq_{n+1})$.

When n is odd (5) must be written

(8) $$kq_{n+2} > a_{n+2}(2p_n + Eq_n),$$

from which it follows that $kq_{n+2} \geq 2p_n a_{n+2}+1$. The lower limit
for the case n odd, that is, $k/(2q_n^2 N^{1/2})$, will obviously be greater
than $a_{n+2}/(q_n q_{n+2})$ provided $2p_n a_{n+2}+1 > 2a_{n+2}q_n N^{1/2}$. And a
moment's calculation, not greatly different from those already
used, shows that this last inequality is certainly satisfied unless
$2a_{n+2} > q_{n+1}$. This is again an exceptional condition, which can
arise only for small values of n. Thus in $2^{1/2} = 1 + \frac{1}{2} + \frac{1}{2} + \cdots$, if
we take the first convergent 1 as the value, we find the error
greater than $1/2^{3/2}$ by the formula $k/(2q_n^2 N^{1/2})$, and greater
than $2/5$ by the formula $a_{n+2}/(q_n q_{n+2})$. In this example the sec-
ond lower limit is better than the first. But this is no longer
true when n is 3 or larger.

It is again true that $N^{1/2}$ must be replaced by a rational ap-
proximation to get a lower limit suitable for computation; this
time the approximation must be too large to insure that we
still have a lower limit, but the values p_{n-1}/q_{n-1}, Nq_n/p_n,
p_{n+1}/q_{n+1}, now have this property. And it is easy to show that
the limit obtained with the aid of the first of these is larger than
$a_{n+2}/(q_n q_{n+2})$ whenever $q_{n-1} > 2a_{n+2}$.

These general proofs that the new limits can only in excep-
tional cases be poorer than the old limits do not begin to show
the advantage possessed by the new limits in most calculations.
This can better be pointed out by taking the difference of the
upper and lower limits in the two cases. If we replace $N^{1/2}$ in the
one limit by the always available approximation Nq_n/p_n, we
find that the difference of our two limits is $k^2/(2p_n q_n^3 N)$. This
difference allows us to estimate at once how accurately we can
approximate with the aid of the nth convergent. The corre-
sponding difference between the usual limits $1/(q_n q_{n+1})$ and
$a_{n+2}/(q_n q_{n+2})$ is $1/(q_{n+1}q_{n+2})$. The ratio of the first difference to
the second is $R = k^2 q_{n+1} q_{n+2}/(2p_n q_n^3 N)$. Making obvious substitu-
tions for q_{n+1} and q_{n+2} in terms of the earlier q's, and remem-
bering that k, as well as each a_i, is less than $2N^{1/2}$, we find
$R < 2(2N^{1/2}+1)^3/(p_n q_n)$. The ratio R is then certainly less than
1 except for small values of p_n and q_n. In fact, R clearly ap-
proaches zero as n increases.

To have one numerical example, consider the approximation
of $2^{1/2}$ by its sixth convergent, $99/70 = 1.414285714+$. The old
limits simply show that this approximation is too large by an

amount larger than 0.0000700 and smaller than 0.0000846. However the new limits, with $N^{1/2}$ in the upper limit replaced by 140/99, tell us that the error is larger than 0.000072150 and smaller than 0.000072158. With this tremendous gain in accuracy we can at once say definitely that $2^{1/2} = 1.41421356$, correct to 8 decimal places.

The limits of the present paper are applicable, under a certain restriction, to a well known method of approximating square roots by iteration. If a_1 is a first approximation to $N^{1/2}$, a second and closer approximation is obtained by taking $a_2 = (a_1 + N/a_1)/2$, a third approximation is formed similarly from the second, and so on. This method has been known since ancient times and recently has been dealt with in a number of papers.* Nevertheless it is interesting to point out that if c and m have the significance of the second footnote of this paper, and if a_1 is taken as p_{mc}/q_{mc}, that is, the mcth convergent to $N^{1/2}$, then a_2 is a later convergent, in fact, the $2mc$th.† Then a_3 is the $4mc$th convergent, and so on. There is, of course, ample justification for using the continued fraction expansion of $N^{1/2}$ to obtain a_1; we naturally want a close approximation without taking an unduly cumbersome fraction, and this is exactly what we obtain by using a convergent. And if we take the additional trouble to compute p_c/q_c, taking $m = 1$ for convenience, we now know that the iterative method continues to give convergents of the form p_{mc}/q_{mc}. Our limits then apply, with $k = 1$, and with n even, provided we form at least a_2. It is worthwhile mentioning that the lower limit $1/(2p_n q_n)$, if we write $a_i = p_n/q_n$, is equal to Bouton's lower limit $(a_{i-1} - a_i)^2/(2a_i)$, and is much easier to use. The proof of the equality is not difficult to make.

Similarly, the upper limit $1/(2q_n^2 N^{1/2})$ is equal to Bouton's upper limit $a_i(a_{i-1} - a_i)^2/(2N)$ in case $N^{1/2}$ is replaced by Nq_n/p_n. Of course if a better approximation is available a better upper limit than Bouton's can be obtained.

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* Bouton, Annals of Mathematics, (2), vol. 10 (1908–9), pp. 167–72; James, American Mathematical Monthly, vol. 31 (1924), pp. 471–5.

† This follows from a theorem which may be found in Chrystal's *Algebra*, Part II, 1889, pp. 440–441. In a note appearing in the American Mathematical Monthly, vol. 39 (1932), pp. 533–535, I have gone into this point in more detail.