and we have

$$\sum G\left(x+z, \frac{y-x}{2}-z, x\right)=\epsilon_1(n) \sum_{r=1}^{T} G(t, 0, 2r-e(t)),$$

where $\epsilon_1(n)=1$ or 0 according as n is or is not a square >0, and $T=[t/2]$.

Similarly, from §4, if $G_1(w, u, v)$ is $G(w, u, v)$ with the restriction of entirety in (u, v) we get

$$4 \sum G_1\left(x+z, \frac{y-x}{2}-z, x\right)$$
$$=\epsilon_1(n)\left[G_1(t, 0, \rho'(t))-G_1(t, 0, \rho'(-t))\right].$$

CALIFORNIA INSTITUTE OF TECHNOLOGY

THE TRANSFORMATION OF LINES OF SPACE BY MEANS OF TWO QUADRATIC REGULI*

BY A. R. WILLIAMS

If we take two quadratic reguli, a line l meets two generators of each. To l we make correspond the other transversal of the four generators. This involutory transformation of the lines of space is one of three, quite similar in principle.† This case admits a very simple and effective algebraic treatment without the use of hyperspace.

We may take for the equations of two non-singular quadrics with real rulings $x_1^2+x_2^2-x_3^2-x_4^2=0$ and $a^2x_1^2+b^2x_2^2-c^2x_3^2-d^2x_4^2=0$. On the former lies the regulus R_1 defined by $x_1-x_3=m(x_4-x_2)$, $x_1+x_3=1/(m(x_4+x_2))$. The Plücker coordinates of a line of this regulus are

$$(1) \quad p_{12}:p_{13}:p_{14}:p_{23}:p_{42}:p_{34}$$
$$=(m^2+1):2m:(m^2-1):(m^2-1):2m:-(m^2+1).$$

The other regulus R_1' on the same quadric is given by the

* Presented to the Society, November 28, 1932.

† Discussed by Mr. J. M. Clarkson in the present issue of this Bulletin, vol. 38, pp. 533–540.

equation $x_1-x_3=m'(x_4+x_2)$, $x_1+x_3=1/(m'(x_4-x_2))$. The coordinates of a line of this regulus are, in the same order,

$$(1')\qquad (m'^2+1):-2m':-(m'^2-1):(m'^2-1):2m':(m'^2+1).$$

Similarly, on the other quadric lies the regulus R_2, a general line of which has the coordinates

$$(2)\ cd(n^2+1):bd\cdot 2n:bc(n^2-1):ad(n^2-1):ac\cdot 2n:-ab(n^2+1);$$

and the regulus R_2', a general line of which has the coordinates

$$(2')\qquad cd(n'^2+1):-bd\cdot 2n':-bc(n'^2-1)$$
$$:ad(n'^2-1):ac\cdot 2n':ab(n'^2+1).$$

To effect the transformation we shall use the reguli R_1 and R_2. That is, to any line l will correspond l', where l' meets the same generators of R_1 and R_2 as l does. A general line p_{ik} meets two generators of R_1 whose parameters are determined by

$$(3)\qquad m^2(-p_{12}+p_{14}+p_{23}+p_{34})$$
$$+2m(p_{13}+p_{42})-p_{12}-p_{14}-p_{23}+p_{34}=0.$$

Similarly, p_{ik} meets two generators of R_2 whose parameters are given by

$$(4)\qquad n^2(-abp_{12}+adp_{14}+bcp_{23}+cdp_{34})+2n(acp_{13}+bdp_{42})$$
$$-abp_{12}-adp_{14}-bdp_{23}+cdp_{34}=0.$$

If any other line p'_{ik} is to meet the same generators of R_1 and R_2, we have, by comparing coefficients, two sets of equations which immediately reduce to

$$(5)\qquad \frac{p'_{12}-p'_{34}}{p_{12}-p_{34}}=\frac{p'_{13}+p'_{42}}{p_{13}+p_{42}}=\frac{p'_{14}+p'_{23}}{p_{14}+p_{23}},$$

$$(6)\qquad \frac{abp'_{12}-cdp'_{34}}{abp_{12}-cdp_{34}}=\frac{acp'_{13}+bdp'_{42}}{acp_{13}+bdp_{42}}=\frac{adp'_{14}+bcp'_{23}}{adp_{14}+bcp_{23}}.$$

Introducing the proportionality factors ρ and σ, we have $p'_{12}-p'_{34}=\rho(p-{}_{12}p_{34})$, $abp'_{12}-cdp'_{34}=\sigma(abp_{12}-cdp_{34})$ and two similar pairs of equations. Solving these equations, we have

$$
(7)\qquad
\begin{aligned}
p_{12}' &= \frac{-\rho cd(p_{12}-p_{34})+\sigma(abp_{12}-cdp_{34})}{ab-cd},\\
p_{34}' &= \frac{-\rho ab(p_{12}-p_{34})+\sigma(abp_{12}-cdp_{34})}{ab-cd},\\
p_{13}' &= \frac{-\rho bd(p_{13}+p_{42})+\sigma(acp_{13}+bdp_{42})}{ac-bd},\\
p_{42}' &= \frac{\rho ac(p_{13}+p_{42})-\sigma(acp_{13}+bdp_{42})}{ac-bd},\\
p_{14}' &= \frac{-\rho bc(p_{14}+p_{23})+\sigma(adp_{14}+bcp_{23})}{ad-bc},\\
p_{23}' &= \frac{\rho ad(p_{14}+p_{23})-\sigma(adp_{14}+bcp_{23})}{ad-bc}.
\end{aligned}
$$

The ratio $\rho:\sigma$ is found from the condition that $p_{12}'p_{34}'+p_{13}'p_{42}'+p_{14}'p_{23}'=0$. This gives $A\rho^2-B\rho\sigma+C\sigma^2=0$ where

$$A=abcd\frac{(p_{12}-p_{34})^2}{(ab-cd)^2}-\frac{(p_{13}+p_{42})^2}{(ac-bd)^2}-\frac{(p_{14}+p_{23})^2}{(ad-bc)^2}$$

and

$$C=\frac{(abp_{12}-cdp_{34})^2}{(ab-cd)^2}-\frac{(acp_{13}+bdp_{42})^2}{(ac-bd)^2}-\frac{(adp_{14}+bcp_{23})^2}{(ad-bc)^2}.$$

Evidently one value of $\rho:\sigma$ is $1:1$. Remembering that we may write $p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$, we verify easily that $A-B+C=0$. Hence the other root is

$$(8)\qquad \rho:\sigma = C:A.$$

Thus the coordinates of p_{ik}' are cubic functions of those of p_{ik}.

Equations (7) show that a line p_{ik} for which $\rho=\sigma\neq 0$ is invariant; that is, it meets two generators of R_1 and two of R_2 which have only one distinct transversal. The condition $\rho=\sigma$ gives the quadratic complex

$$(9)\qquad -\frac{abp_{12}^2-cdp_{34}^2}{ab-cd}+\frac{acp_{13}^2+bdp_{42}^2}{ac-bd}+\frac{adp_{14}^2+bcp_{23}^2}{ad-bc=0}.$$

The generators of R_1 and R_2 belong to this complex. To a line p_{ik} for which $\rho=0$ but $\sigma\neq 0$ corresponds a line whose coordinates, down to a constant factor, are

$$p'_{12} = p'_{34} = \frac{abp_{12} - cdp_{34}}{ab - cd},$$

$$p'_{13} = -p'_{42} = \frac{acp_{13} + bdp_{42}}{ac - bd},$$

$$p'_{14} = -p'_{23} = \frac{adp_{14} + bcp_{23}}{ad - bc}.$$

Since $-p'_{12} + p'_{14} + p'_{23} + p'_{34} = 0$, $p'_{13} + p'_{42} = 0$, and $-p'_{12} - p'_{14} - p'_{23} + p'_{34} = 0$, this line by equation (3) meets all the generators of R_1; that is, it is itself a generator of R_1'. It is geometrically obvious that if p_{ik} meets two generators of R_2 which are met by the same generator of R_1', the latter will be p_{ik}'. Any generator of R_1' meets two generators of R_2; and no two generators of R_1' meet the same two generators of R_2 if the quadrics are not specially related. Hence the complex $\rho = 0$ is made up of ∞^1 congruences, each congruence consisting of the lines that meet a pair of generators of R_2 that are met by the same generator of R_1'. Each generator of R_2 is a member of two such pairs. Similarly, to a line p_{ik} for which $\sigma = 0$ but $\rho \neq 0$ corresponds a generator of R_2'; and the complex $\sigma = 0$ consists of ∞^1 congruences, each composed of the lines that meet a pair of generators of R_1 that are met by the same generator of R_2'. The generators of both R_1' and R_2' satisfy $\rho = 0$ and $\sigma = 0$. That is, they belong to the congruence of lines that do not have unique transforms. Such lines are easily found. Through P, any point on the intersection of the two quadrics, pass g_1, a generator of R_1, and g_2, a generator of R_2; also g_1' and g_2' generators of R_1' and R_2' respectively. The plane of g_1' and g_2' contains f_1 and f_2, generators of R_1 and R_2 respectively. Thus any line through P in this plane meets the four generators g_1 and f_1 of R_1, and g_2 and f_2 of R_2. Analytically, if we take a_{ik} and b_{ik}, any two generators of R_1' and R_2' that are supposed to meet, see (1′) and (2′), we find easily that the coordinates of any line of their pencil $a_{ik} + \lambda b_{ik}$ satisfy $\rho = \sigma = 0$. Thus the congruence is composed of ∞^1 flat pencils, each determined by a generator of R_1' and a generator of R_2'. Each such generator belongs to two of these pencils. If two generators of R_1 and two of R_2 were linearly dependent, they would lie all four on the same proper hyperboloid, or three of them would belong to the same flat pencil. In either case they would have an infinite number of transversals and no one of these transversals would then have a unique transform. This cannot happen, however, unless there is some special relation between the two quadrics. For if, using (1) and (2), we let m_1 and m_2 be the parameters of two generators of R_1, and n_1 and n_2 the parameters of two generators of R_2, linear dependence would necessitate the simultaneous vanishing of all the four-rowed determinants of a matrix 6 by 4. It turns out that this can not be accomplished by variation of the parameters alone.

The characteristics of the transformation can be inferred from the above. To a ruled surface of order n corresponds one of order $3n$. For each line of the congruence $\rho=\sigma=0$, that is not at the same time a generator of R_1' or R_2', contained by a ruled surface, the order of the transform will be reduced by one. Each generator of R_1' or R_2' will reduce the order by two. Thus a generic flat pencil contains two invariant lines, two lines belonging to $\rho=0$, and two belonging to $\sigma=0$. Hence, the corresponding cubic surface contains two generators of R_1' and two of R_2', and their presence agrees with the fact that to the cubic surface corresponds the original pencil. In this connection an interesting fact may be noted. If a_{ik} and b_{ik} are two lines meeting at O, any line of their pencil is given by $a_{ik}+\lambda b_{ik}$. Since the coordinates of the transform of the latter contain λ to the third degree, it will for all values of λ meet two fixed lines, generally distinct, whose coordinates are functions of a_{ik} and b_{ik}. One of these must pass through O and the other must lie in the plane of the pencil. Since the pencil contains two invariant lines, the former, passing through O, is the double line of the cubic surface and the latter the directrix. Calling the double line r and the directrix r', it is easy to see that to any plane pencil whose center is on r and whose plane passes through r' corresponds a similar cubic surface whose double line is r and directrix r'. This relation is reciprocal. To any plane pencil whose center is on r' and whose plane passes through r corresponds a ruled cubic surface whose double line is r' and whose directrix is r. That is, if any line meets r and r', its transform by (7) does so also, and the congruence determined by r and r' is invariant as a whole. Here we have, as it were, a transformation within a transformation. Taking a general line for r, is there a unique r' corresponding to it in this sense? To answer this, let r be determined by the points

(y) and (z) so that $r_{ik}=y_i z_k - y_k z_i$. Then we find that in addition to the quadric cone of invariant lines whose center is (y) there is a flat pencil of lines with center at (y) whose transforms by (7) meet r. There is a similar flat pencil at (z) and at every other point of r. The planes of these flat pencils form an axial pencil whose axis is r'. Its coordinates are easily found in terms of the coordinates of r and prove to be just the coordinates of the line which would correspond to r by the transformation exactly similar to (7) which is effected by means of the supplementary reguli R_1' and R_2'. As we should expect, this latter transformation admits the same invariant complex as the original. In fact, if r in the above discussion is an invariant line under (7), the flat pencil of lines at any point of r whose transforms by (7) meet r contains r. To such a flat pencil corresponds a ruled cubic surface of which r is both double line and directrix.

To a quadratic regulus corresponds a ruled sextic. If the regulus contains for example two generators of R_1' and one of R_2', the order of the corresponding surface should reduce to nothing. This is obviously correct. For if the intersection of two quadrics consists in part of two skew lines, the remainder must be two skew lines of the other ruling. So all the rulings of the regulus in question, including the generator of R_2' which belongs to it, meet the same two rulings of R_1. Therefore, to the whole regulus corresponds the generator of R_2'. To a linear complex corresponds a cubic complex which contains the generators of R_1' and R_2' each counted ∞^1 times. For consider a pair of generators of R_2 that are met by the same generator of R_1'. The lines of the linear complex that meet this pair of generators constitute a quadratic regulus, and to every line of it corresponds the generator of R_1'. To such a cubic complex corresponds the complex of order 9 made up of ρ^2, σ^2, and the original linear complex. Finally, it should be noted that the order of the complex corresponding to a complex that contains R_1' or R_2' is lowered by 2. Thus the harmonic complex, that is, the quadratic complex of lines that meet the two quadrics in four harmonic points, contains both R_1' and R_2', and to it corresponds another quadratic complex. Of course, R_1 and R_2 also belong to the harmonic complex; and R_1' and R_2' belong to its transform by (7); but R_1 and R_2 do not.

THE UNIVERSITY OF CALIFORNIA