NOTE ON A THEOREM DUE TO BROMWICH

BY H. L. GARABEDIAN

The following well known theorem is due to Bromwich.*

THEOREM. *Suppose* (i) *that the series* $\sum a_n$ *is summable by Cesàro means of order* k *to the sum* s, (ii) *that* v_n *is a function of* x *with the properties*

$$\left.\begin{array}{ll}(\alpha) & \sum n^k \left| \Delta^{k+1} v_n \right| < K\dagger \\ (\beta) & \lim\limits_{n\to\infty} n^k v_n = 0\end{array}\right\} \quad if \quad x > 0,$$

$$(\gamma) \qquad \lim_{x\to 0} v_n = 1,$$

where K *is independent of* x *and* n. *Then the series* $\sum a_n v_n$ *converges if* x *is positive, and*

$$\lim_{x\to 0} \sum a_n v_n = s.$$

I propose to establish this theorem by a more direct and shorter method than that used by Bromwich. Moreover, this proof affords a method of exhibiting a k-fold summability with infinite matrix of reference, analogous to well known definitions of summability with finite matrices of reference which make use of repeated means, for any v_n which satisfies the conditions of the theorem under discussion.

By hypothesis the series $\sum a_n$ is summable by Cesàro means of order k, so that if

$$S_n^{(k)} = \binom{n+k-1}{k-1} s_0 + \binom{n+k-2}{k-1} s_1 + \cdots + \binom{k-1}{k-1} s_n$$

and

$$A_n^{(k)} = \binom{n+k}{k},$$

* Mathematische Annalen, vol. 65 (1907–08), pp. 350–369; p. 359.

† Since all of the terms in the series $\sum n^k \left|\Delta^{k+1} v_n\right|$ are positive, this condition implies the convergence of the series.

then

$$C_n^{(k)} = \frac{S_n^{(k)}}{A_n^{(k)}}$$

has a definite limit s as n tends to ∞. We may also define $S_n^{(k)}$ by means of the identities

$$\sum S_n^{(k)} x^n = (1-x)^{-k} \sum s_n x^n = (1-x)^{-(k+1)} \sum a_n x^n,$$

from which it follows that

$$\sum s_n x^n = (1-x)^k \sum S_n^{(k)} x^n$$

and

$$\sum a_n x^n = (1-x)^{k+1} \sum S_n^{(k)} x^n.$$

It results at once from the last identity that

$$\begin{aligned} a_n = S_n^{(k)} - \binom{k+1}{1} S_{n-1}^{(k)} + \binom{k+1}{2} S_{n-2}^{(k)} - \cdots \\ + (-1)^{k+1} \binom{k+1}{k+1} S_{n-k-1}^{(k)}, \end{aligned} \tag{1}$$

where it is understood that when a negative subscript occurs in the formula, the corresponding $S^{(k)}$ is to be replaced by zero.

Now, we form the series

$$F(x) = \sum_{n=0}^{\infty} a_n v_n(x),$$

or, using (1),

$$\begin{aligned} F(x) = \sum_{n=0}^{\infty} \Bigg[S_n^{(k)} - \binom{k+1}{1} S_{n-1}^{(k)} + \binom{k+1}{2} S_{n-2}^{(k)} - \cdots \\ + (-1)^{k+1} \binom{k+1}{k+1} S_{n-k-1}^{(k)} \Bigg] v_n(x). \end{aligned} \tag{2}$$

We are justified by the conditions (ii) in ordering the terms of (2) with respect to $S_n^{(k)}$ to get

$$F(x) = \sum_{n=0}^{\infty} S_n^{(k)} \left[v_n(x) - \binom{k+1}{1} v_{n+1}(x) + \cdots \right.$$
$$\left. + (-1)^r \binom{k+1}{r} v_{n+r}(x) + \cdots + (-1)^{k+1} v_{n+k+1}(x) \right],$$

or

$$(3) \qquad F(x) = \sum_{n=0}^{\infty} \binom{n+k}{k} \Delta^{k+1} v_n(x) C_n^{(k)}.$$

Since $\lim_{n\to\infty} C_n^{(k)} = s$, it remains to show that the method of summation with infinite matrix of reference defined by (3) is *regular*,* which is to say that $\lim_{x\to 0} F(x) = s$. Accordingly, we must require in the present case that

$$(a) \qquad \lim_{x\to 0} \binom{n+k}{k} \Delta^{k+1} v_n(x) = 0 \text{ for every } n,$$

$$(b) \qquad \lim_{x\to 0} \sum_{n=0}^{\infty} \binom{n+k}{k} \Delta^{k+1} v_n(x) = 1,$$

$$(c) \qquad \sum_{n=0}^{\infty} \binom{n+k}{k} \left| \Delta^{k+1} v_n(x) \right| < K$$

for every $x > 0$, K independent of x.

It follows from condition (α) of the hypotheses that

$$\lim_{x\to 0} \binom{n+k}{k} \Delta^{k+1} v_n(x) = \binom{n+k}{k} \Delta^{k+1} 1 = 0.$$

Accordingly, the requirement (a) is satisfied.

Now, we need the identity

$$(4) \qquad \begin{aligned} \sum_{n=0}^{n} \binom{n+k}{k} \Delta^{k+1} v_n(x) &= v_k(x) + \sum_{\nu=0}^{k-1} \frac{k!}{(k-\nu)!\nu!} \Delta^{k-\nu} v_\nu(x) \\ &\quad - \sum_{\nu=0}^{k} \frac{(n+k-\nu)!}{(k-\nu)!n!} \Delta^{k-\nu} v_{n+1}(x). \dagger \end{aligned}$$

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* See Carmichael, *The theory of summable series*, this Bulletin, vol. 25 (1918–19), pp. 97–131; p. 117.

† See H. L. Garabedian, Annals of Mathematics, vol. 32 (1930), pp. 83–106; p. 91.

As n tends to infinity every term involving n on the right-hand side of (4) tends to zero by virtue of condition (β) of the hypotheses. It follows that

$$(5)\quad \sum_{n=0}^{\infty}\binom{n+k}{k}\Delta^{k+1}v_n(x) = v_k(x) + \sum_{\nu=0}^{k-1}\frac{k!}{(k-\nu)!\nu!}\Delta^{k-\nu}v_\nu(x).$$

Moreover, by virtue of condition (α),

$$(6)\quad \lim_{x\to 0}\sum_{\nu=0}^{k-1}\frac{k!}{(k-\nu)!\nu!}\Delta^{k-\nu}v_\nu(x) = 0.$$

Accordingly, from (5), (6), and (α) we have

$$\lim_{x\to 0}\sum_{n=0}^{\infty}\binom{n+k}{k}\Delta^{k+1}v_n(x) = \lim_{x\to 0} v_k(x) = 1,$$

and the requirement (b) is fulfilled.

Finally, we note that the expression

$$\sum_{n=0}^{\infty}\binom{n+k}{k}\left|\Delta^{k+1}v_n(x)\right|$$

will be uniformly bounded or fail to be uniformly bounded according as the expression $\sum_{n=0}^{\infty}n_k\left|\Delta^{k+1}v_n(x)\right|$ is uniformly bounded or fails to be uniformly bounded. It is understood in this statement that x is restricted to positive values. Hence, by condition (γ), the last of the requirements (c) for regularity is satisfied.

We conclude that

$$\lim_{x\to 0}F(x) = \lim_{x\to 0}\sum_{n=0}^{\infty}\binom{n+k}{k}\Delta^{k+1}v_n(x)C_n^{(k)} = s.$$

We exhibit in the function

$$\phi(n, x) = \binom{n+k}{k}\Delta^{k+1}v_n(x)$$

a convergence factor which, associated with a method of summation with infinite matrix of reference, affords a method of constructing a k-fold method of summability with infinite matrix of reference for any $v_n(x)$ which satisfies the conditions of Bromwich's theorem. Examples of functions $v_n(x)$ which satisfy these requirements* are the LeRoy convergence factor:

* H. L. Garabedian, loc. cit.

$$v_n(x) = \frac{\Gamma[(1 - x)n + 1]}{\Gamma(1 + n)};$$

the Mittag-Leffler convergence factor:

$$v_n(x) = \frac{1}{\Gamma(1 + nx)};$$

and the Dirichlet series convergence factors:

$$v_n(x) = e^{-\lambda(n)x},$$

where $\lambda(n)$ must be a logarithmico-exponential function of n which tends to infinity with n but not as slowly as $\log n$ nor faster than n^Δ, where Δ is any constant however large.

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A THEOREM ON SYMMETRIC DETERMINANTS

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1. *Introduction.* In a recent paper* the writer proved the following theorem.

If $D = |a_{ij}|$ is a real symmetric determinant of order n, $n > 5$, in which $a_{ii} = 0$, $(i = 1, 2, \cdots, n)$, and M is any principal minor of D of order $n-1$, then if all fourth order principal minors of M are zero, D vanishes.

The purpose of the present note is to establish a second theorem of a similar nature which applies to complex as well as to real determinants. It will be shown also that when a_{ij}, $(i \neq j)$, $(i,j = 1, 2, \cdots, n)$, is real and different from zero the conditions of this second theorem imply those of the above.

2. *A Second Theorem.* The theorem with which this note is concerned may be stated as follows.

THEOREM. *If $D = |a_{ij}|$ is a symmetric determinant of order n, $n > 5$, in which $a_{ii} = 0$, $(i = 1, 2, \cdots, n)$, and M is any principal minor of D or order $n-1$, then if all fourth order principal minors of D, which are not minors of M, are zero, D vanishes.*

* This Bulletin, vol. 38 (1932), p. 259.