SUMMATION OF FOURIER SERIES*

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1. *Introduction.* My topic, the summation of Fourier series, has been dealt with in addresses delivered to this Society by C. N. Moore, G. H. Hardy, and again by C. N. Moore, the last one less than two years old.† If I venture to speak on the same subject again, it is because the field is in a state of brisk and steady development so that it is possible for me to deal with matters which have not been exhausted by previous speakers. In particular, I am happy to be able to include in my report a number of results, published and unpublished, found by J. D. Tamarkin and myself during the last few years.

The term Fourier series is used in at least five different senses in the current literature. In the present report the term signifies a trigonometrical Fourier-Lebesgue series, that is, a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1}$$

or, in the complex form preferred nowadays,

$$\sum_{n=-\infty}^{+\infty} f_n e^{nix} \tag{2}$$

where the coefficients are determined by

$$\left.\begin{matrix} a_n \\ b_n \end{matrix}\right\} = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} nt\,dt, \quad f_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)e^{-nit}dt \tag{3}$$

respectively. Here $f(x)$ is supposed to be integrable in the sense of Lebesgue in the interval $(-\pi, +\pi)$, and is extended by the convention $f(x+2\pi) = f(x)$ outside this interval.

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† The addresses of C. N. Moore have appeared in this Bulletin, vol. 25 (1918–19), pp. 258–276, and vol. 37 (1931), pp. 240–250.

These formulas associate a formal series with every integrable function $f(x)$, the *Fourier series* of $f(x)$. It is natural to ask in what sense the series represents or determines the function. If we write

$$(4)\qquad s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) = \sum_{k=-n}^{+n} f_k e^{kix},$$

our first guess is of course that

$$(5)\qquad \lim_{n\to\infty} s_n(x) = f(x).$$

All the evidence accumulated during a century indicates, however, that this relation must be considered as an exception rather than as the rule, though a quantitative measure of the improbability of (5) is still lacking.

Ordinary convergence being a clear failure in the study of Fourier series, it is obviously necessary to consider some notion of generalized convergence instead. Essentially two different lines of attack are available here. We may use the modern theory of systems of integrable functions involving the concepts of strong convergence, weak convergence, convergence in measure and so on. Or we may use the theory of transformations of sequences together with the related theory of singular integrals. It is with the second mode of procedure that I shall be concerned here. I want to emphasize, however, that these two methods are closely interwoven, and that both are indispensable for a complete theory of Fourier series.

The early history of summation of Fourier series has been outlined often enough. Let me merely recall here that the theory received its first strong impetus from the fundamental paper of Fejér in 1903 [11*; preliminary communications go back to 1900] which was followed by the investigations of Lebesgue in 1905–09 [26, 27, 28]. These memoirs form the point of departure of the modern theory of Fourier series. They also kindled a wide interest among analysts in the theory of summability which had already proved its usefulness in connection with the problem of analytic continuation around 1900 [see, for example, 3, 29, 30, 31].

* Reference to the Bibliography at the end of this paper.

2. *Effectiveness*. Throughout the development of the theory of summation of Fourier series we may discern two different points of view, a local one and a global one, which occasionally are in conflict but usually complete each other.

The phenomena of convergence or summability of a Fourier series are of a purely local character. Only the properties of the function in the immediate neighborhood of a point determine whether or not its Fourier series is summable at that point by a particular definition of summability. We should like to know what local properties of the function imply such summability of its Fourier series, and, conversely, what properties of the function are implied by its Fourier series being so summable. We have here two types of limiting processes, one referring to the function, the other referring to its Fourier series, and the question is when the existence of one limit implies that of the other.

This is the local problem, but there is also a problem in the large. Every function $f(x) \subset L$ determines a unique Fourier series, but to a given Fourier series correspond infinitely many functions the difference of any two of which is a null function. One of the main problems of our theory is how to pass from the series to the function, or, rather, to any one of the set of equivalent functions. This requires that we be able to assign a sum to the series for almost all values of x, and that the sum-function has the given series as its own Fourier series. The global point of view thus leads to the question: *Does a given definition of summability sum Fourier series to "the correct sum" almost everywhere?* The characterization of all such definitions then becomes a problem of interest and importance both to the theory of Fourier series and to the theory of summability.

But one can also occupy a more general point of view of which those already mentioned are only special aspects. Let there be given a class C of series

$$\sum_{-\infty}^{+\infty} c_n u_n(x), \tag{6}$$

the terms of which are defined for $a \leqq x \leqq b$. Suppose further that with each series of C there is associated a function $f(x)$ defined for almost all x in (a, b). To fix our ideas, we may think of the class L of all Fourier-Lebesgue series, $f(x)$ being in this case one of the equivalent functions which give rise to the particular series. But we could also let C be the class of conjugate series, or of Fourier-Denjoy series, or of Fourier series in the sense in which this term is used in the theory of almost periodic functions, or of Legendre series etc. In each case it is possible to make the required association between series and function. With each such function $f(x)$ we associate a set E_f of points in (a, b) where $f(x)$ has a finite definite value and satisfies a prescribed condition of regularity. The latter condition will usually imply that $f(x)$ and (or) some related function are continuous in some sense or other. We can think of ordinary continuity or continuity in the mean or strong continuity (a Lipschitz condition or a similar one). Finally, let a definition of summability be given, A say, which associates with the series (6) a sequence or a one-parameter family of functions $T(x, \omega; f(\cdot), A)$. If now

$$\lim_{\omega\to\infty} T(x, \omega; f(\cdot), A) = f(x) \tag{7}$$

for every series in C and for every x in E_f, we say that *the definition A is (C, E_f)-effective*. The notation indicates that this definition can be used for the summing of any series in C, and will assign as the sum of the series the associated function $f(x)$ for every $x \subset E_f$. It should be realized that E_f may be vacuous and that we disregard entirely what happens in the complementary set $(a, b)-E_f$.

In (7) we presuppose convergence in the sense of Cauchy, but we can reach still greater generality by considering various forms of generalized convergence, e.g., convergence in the mean, in which case the definitions have to be modified in an obvious manner. (For the development of this terminology and this point of view, see Hille and Tamarkin [20], third and fourth notes, [21] and [22].) A few examples will clarify the notation.

(i) Let C be the class L of all Fourier-Lebesgue series, $f(x)$ any one of the equivalent functions giving rise to the series. Let E_f be the set of points where $f(x)$ is continuous. We say that A is *(F)-effective* if it has this type of (C, E_f)-effectiveness.

(ii) Let C and $f(x)$ be defined as above and let E_f be the set of points where

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{+h} | f(x+t) - f(x) | \, dt = 0. \tag{8}$$

We designate this type as *(L)-effectiveness*.

Thus A is (F)-effective if it sums the Fourier series of $f(x)$ to the sum $f(x)$ at all points of continuity of $f(x)$, and (L)-effectiveness implies summability to the sum $f(x)$ almost everywhere. It is clear that every (L)-effective definition is also (F)-effective, but the converse is scarcely likely to be true. There exist definitions of summability which are (F)-effective, and which have not been proved to be (L)-effective so far.

In the following I shall be concerned chiefly with the problem of what definitions of summation are (F)- or (L)-effective. Before leaving these general considerations I want to emphasize, however, that this is only a particular case of the general question of (C, E_f)-effectiveness, though one of the most important cases. It is customary to consider various associated series simultaneously with the theory of Fourier series proper such as *the conjugate* (=allied) *series*, *the derived series of functions of bounded variation* and their *conjugate series*. Without attempting to be systematic, I shall frequently mention what is known about the corresponding effectiveness problems for such classes of series.

3. *Conditions of Effectiveness*. We pass over to the question of how to decide if a given definition of summation A is (F)- or (L)-effective. A is supposed to associate with $f(x) \subset L$ a family of functionals $T(x, \omega; f(\cdot), A)$. A particularly simple example is that in which A is a regular linear sequence-to-sequence transformation with finite reference defined by a triangular matrix $\|a_{m,n}\|$ and the equations

$$y_m = \sum_{n=0}^{m} a_{mn}x_n, \qquad (m = 0, 1, 2, \cdots). \tag{9}$$

Substituting

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x+t)\frac{\sin\,(n+\frac{1}{2})t}{\sin\,\frac{1}{2}t}\,dt \tag{10}$$

for x_n, we get

$$T(x, m; f(\cdot), A) = \int_{-\pi}^{+\pi} f(x+t)K_m(t)dt, \tag{11}$$

where

$$K_m(t) = \sum_{n=0}^{m} a_{mn} \frac{\sin (n + \frac{1}{2})t}{2\pi \sin \frac{1}{2}t} . \tag{12}$$

A necessary and sufficient condition that

$$T(x, m; f(\cdot), A) \to f(x) \text{ as } m \to \infty \tag{13}$$

at every point of continuity of $f(x)$ is given by

$$\int_0^\pi | K_m(t) | dt < M \tag{14}$$

for a fixed finite M independent of m. The representation (11) and condition (14) remain valid under much more general assumptions on A. But if we merely assume that $T(x, \omega; f(\cdot), A)$ is a linear functional of $f(\cdot)$ which as a function of x belongs to L for every finite ω and which converges to $f(x)$ at every point of continuity as $\omega \to \infty$, then the integrals in (11) and (14) have to be replaced by suitable Stieltjes integrals.

No necessary and sufficient conditions for (L)-effectiveness seem to be known. For a rather wide class of definitions, including all regular definitions of finite reference, a set of sufficient conditions is given by (14) together with

$$\lim_{m \to \infty} K_m(t) = 0, \ t \neq 0, \tag{15}$$

$$\int_0^\delta | d_t[tK_m(t)] | < M, \ \delta < \delta_0, \ m_0 < m. \tag{16}$$

It should be mentioned, however, that condition (16) is apt to fail in all except the very simplest cases occurring in the theory of Fourier series.

Our problem can also be approached through *the relations of relative inclusion* known to exist between various definitions of summability. If every series summable A_2 is also summable A_1 to the same sum, we say, following W. A. Hurwitz, that the definition A_1 includes the definition A_2. Hence, *if A_1 includes A_2 and if A_2 is (C, E_f)-effective, so is A_1.* Now Cesàro's definition of order $\alpha > 0$ is known to be (L)-effective (G. H. Hardy [16]; many other proofs are known). It follows that *any definition A which includes a Cesàro definition of positive order is necessarily (L)-effective.* Incidentally, such a definition will also sum the conjugate series, the derived series of a function of bounded variation and its conjugate series, each to its proper sum for almost all values of x. The value of this and of similar criteria is limited by the fact that (L)-effective definitions are not necessarily comparable or even consistent, nevertheless such criteria will frequently be used in the discussion below.

4. *Plan of Survey*. I proceed to the main part of this address which is devoted to a preliminary census of the known definitions of summability from the point of view of (F)- and (L)-effectiveness. In this short exposition I can obviously only consider some of the more important definitions where the actual choice of course is largely dictated by my own temporary interests. The classification of definitions of summability is not far developed. I use the customary grouping according to finite or infinite reference; a third heading is added for summation of integrals.

I. Definitions with Finite Reference

I consider four separate types of definitions of summability based upon a single generating sequence (§§5–8) and a fifth one based upon two generating sequences (§9). The discussion of means of the closed cycle which may be of finite or infinite reference starts in §10 and is continued in §11 of II. All these six types contain the definitions of Cesàro as special cases.

5. *Retrogressive Means*. We employ a sequence of complex numbers $\{p_\nu\}$ such that

$$P_n \equiv p_0 + p_1 + \cdots + p_n \neq 0, \tag{17}$$

and take as the generalized limit of the sequence $\{s_n\}$ the expression

$$(N, p_\nu) - \lim s_n = \lim_{n\to\infty} P_n^{-1}(p_n s_0 + p_{n-1}s_1 + \cdots + p_0 s_n) \tag{18}$$

if this limit exists. The conditions for regularity are

$$\sum_{k=0}^{n} | p_k | < C | P_n | , \; p_n/P_n \to 0. \tag{19}$$

It is customary to refer to these means as those of Nörlund [36] though the first discovery goes back to Woronoi [54]. The term "retrogressive" is used here for the purpose of comparison with the means of §6, and refers to the manner in which the weights $\{p_\nu\}$ are attached to the elements of the sequence $\{s_n\}$.

The relative inclusion theory of these means has been studied by M. Riesz [43]. The condition that (N, p_ν) shall include $(N, 1)=(C, 1)$ is particularly simple, viz.

$$(20) \qquad n\,|\,p_0\,| + \sum_{k=1}^{n}(n-k+1)\,|\,p_k - p_{k-1}\,| < C\,|\,P_n\,|\,,$$

and is satisfied, for example, if p_n is positive and monotone increasing. Any such definition (N, p_ν) is consequently (L)-effective. A direct study of the effectiveness of these means with respect to Fourier series has been made by Hille and Tamarkin ([20], first note, [21]) who found that the following conditions are sufficient for (L)-effectiveness:

$$(21) \qquad n\,|\,p_n\,| < C\,|\,P_n\,|\,,$$

$$(22) \qquad \sum_{k=1}^{n} k\,|\,p_k - p_{k-1}\,| < C\,|\,P_n\,|\,,$$

$$(23) \qquad \sum_{k=1}^{n} \frac{|\,P_k\,|}{k} < C\,|\,P_n\,|\,.$$

If these conditions are satisfied the method (N, p_ν) will also sum the conjugate series, the derived series of a function of bounded variation, and its conjugate series to the proper sums almost everywhere. If $p_n > 0$ and satisfies (21) and (22), thus in particular if p_n is ultimately monotone decreasing, then (23) is a necessary as well as sufficient condition for (F)-effectiveness. A simple example of a definition which is not (F)-effective is given by the harmonic mean, $(N, (\nu+1)^{-1})$.

6. *Progressive Means*. We again employ a sequence $\{p_\nu\}$ satisfying (17), but the generalized limit is taken to be

$$(24) \qquad (R, p_\nu) - \lim s_n = \lim_{n\to\infty} P_n^{-1}(p_0 s_0 + p_1 s_1 + \cdots + p_n s_n)$$

instead. The conditions of regularity are now

$$(25) \qquad \sum_{k=0}^{n} |\,p_k\,| < C\,|\,P_n\,|\,,\ P_n \to \infty\,.$$

We can refer to these weighted means as progressive means. If $p_n > 0$ they become a particular case of the discrete Riesz means $R(P_n, 1)$. The choice $p_\nu = 1$ gives $(R, 1) = R(n, 1) = (C, 1)$. Except for particular instances, the question of effectiveness of such means does not seem to have been studied, but there is plenty of information available from the inclusion theory. Thus G. H. Hardy [15] has proved that any series summable (R, p_ν), $p_\nu > 0$, is summable $(R, p_\nu\, \epsilon_\nu)$ to the same sum if the latter definition is regular and ϵ_ν is monotone decreasing. It follows in particular that (R, ϵ_ν), where $\sum \epsilon_\nu$ is divergent, is always an (L)-effective definition. As a special case we note that the logarithmic mean $(R, (\nu+1)^{-1})$ is (L)-effective whereas we have seen that the harmonic mean $(N, (\nu+1)^{-1})$ is not even (F)-effective. (For a recent study of the application of the logarithmic mean to Fourier series, see Hardy [17].) Necessary and sufficient conditions that (R, p_ν) include $(R, 1) = (C, 1)$ are simply conditions (21) and (22) of §5, which are satisfied if, e.g., p_n is a monotone increasing function of n which does not grow faster than every power of n. Any such mean is (L)-effective.

7. *Typical Means*. In order to be in agreement with standard notation we put $P_n = \lambda_{n+1}$, and suppose $p_n = \lambda_{n+1} - \lambda_n > 0$, $n \geqq 0$. The typical means of M. Riesz [18, 41] of type λ, order κ, of the first kind, (R, λ, κ), involve taking as the generalized sum of the series $\sum u_n$ the expression

$$\lim_{\omega \to \infty} \sum_{\lambda_n < \omega} \left[1 - \frac{\lambda_n}{\omega}\right]^\kappa u_n \tag{26}$$

when this limit exists. Here ω is a continuous parameter. If, however, $\omega \to \infty$ through a denumerable sequence of values, usually through the set $\{\lambda_n\}$ itself, we get various forms of discrete Riesz means, $R(\lambda, \kappa)$. We use the latter notation exclusively for the case in which ω is restricted to the set $\{\lambda_n\}$. We have already observed that $R(\lambda, 1) = (R, \lambda_{\nu+1} - \lambda_\nu)$ and it is easy to see that $R(n, \kappa) = (N, (\nu+1)^\kappa - \nu^\kappa)$. Since every series summable (R, n, κ) is also summable (R, λ, κ) to the same sum if λ_n is a logarithmico-exponential function of n which does not tend to infinity faster than every power of n [18], it follows that any such definition (R, λ, κ) is (L)-effective.

Applications of various types of typical means to the theory of Fourier series are due to, among others, S. Chapman [7, 8], M. Riesz [41], M. H. Stone [48], and W. H. Young [55, 56].

8. *Momental Means*. We designate the generating sequence by $\{\mu_n\}$. The μ's are supposed to be moments of a mass distribution over the interval (0, 1). To be precise, we assume the existence of a function $q(u)$ with the following properties:

(27) $q(u)$ *is of bounded variation in* (0, 1);

(28) $q(u)$ *is continuous at* $u=0$ *and* $q(0)=0$;

(29) $q(1)=1$;

$$\mu_n = \int_0^1 u^n d[q(u)], \qquad (n = 0, 1, 2, \cdots). \tag{30}$$

The generalized limit of the sequence $\{s_n\}$ is taken to be

$$\lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} s_k \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} \mu_{j+k} \tag{31}$$

if this limit exists. According to Hausdorff [19] every such function $q(u)$ gives rise to a regular definition of summability which we denote by $[H, q(u)]$. All such definitions are known to be mutually consistent, and Cesàro's definitions of positive order belong to the set. The *analytically regular definitions* of Hurwitz and Silverman [23] also belong to this class.

The theory of relative inclusion of momental means offers a fascinating study; particular results found by Hausdorff [19] and by Hille and Tamarkin [unpublished] throw an interesting light over the question of the effectiveness of these means with respect to Fourier series. A direct attack on the latter question has been made by Hille and Tamarkin [20, details unpublished] who have shown that the properties of the Fourier transforms of $q(u)$ are decisive for this problem. Put

$$C(v) = \int_0^1 [q(u) - u] \cos uv \, du, \tag{32}$$

$$J(\epsilon) = \int_\epsilon^1 \frac{Q(u)}{u} du + \int_0^{1-\epsilon} \frac{Q(1) - Q(u)}{1 - u} du, \tag{33}$$

$$Q(u) = \int_0^u | dq(s) | .$$

The condition

$$\int_0^\infty |C(v)|\,dv < \infty \tag{34}$$

is necessary for the (F)-effectiveness of the definition $[H, q(u)]$; and (34) together with

$$\lim_{\epsilon\to 0} J(\epsilon) < \infty \tag{35}$$

is sufficient.

Condition (34) is of a type which we shall encounter again in §§10 and 11. It expresses the fact that *the Fourier cosine transform of a certain kernel is integrable over the range* $(-\infty, +\infty)$; it imposes restrictions on the kernel analogous to those required for *the absolute convergence of a Fourier series.* Pursuing this analogy, Hille and Tamarkin [20, third note] have announced a number of conditions, necessary and sufficient or merely sufficient, that the Fourier transform $G(u)$ of a function $g(u)$ shall have this property. Among these conditions the following may be emphasized:*

1. $G(u) \subset \mathfrak{L}_1$, *if* (i) $g(u) \subset \mathfrak{L}_p$ *for some* p, $1<p\leqq 2$, (ii) $g(u)$ *is of bounded variation in* $(-\infty, +\infty)$, *and* (iii) $g(u)$ *is continuous in* $(-\infty, +\infty)$, *its modulus of continuity*, $\omega(h)$, *satisfying the condition*

$$\int_0^1 [\omega(h)]^{1/p'} h^{-1} dh < \infty . \tag{36}$$

2. $G(u) \subset \mathfrak{L}_1$, *if* (i) $g(u) \subset \mathfrak{L}_p$ *for some* p, $1<p\leqq 2$, (ii) $g(u)$ *is absolutely continuous and* $g'(u) \subset \mathfrak{L}_1$, *and* (iii)

$$\int_0^1 \Omega(h) h^{-1} dh < \infty ,$$

* We denote by $\mathfrak{L}_p$, $1\leqq p\leqq 2$, the class of functions which are measurable and the pth power of the absolute values of which are integrable over the interval $(-\infty, +\infty)$. The conjugate exponent of $p>1$ is denoted by p', that is, $1/p+1/p'=1$. The Fourier transform referred to is the complex one; the cosine transform is obtained if $g(-u)=g(u)$. In applying the results to momental means we take $g(u)=q(u)-u$ for $0\leqq u\leqq 1$ and put $g(u)=0$, $u>1$, $g(u)=g(-u)$, $u<0$. A similar remark, mutatis mutandis, applies in the case of the means of the closed cycle below.

(37)
$$\Omega(h) = \int_{-\infty}^{+\infty} | g'(u+h) - g'(u-h) | \, du.$$

On the basis of these and similar criteria it is possible to construct mass functions $q(u)$ which give rise to (F)-effective means. It should be noted that it is neither necessary nor sufficient that $q(u)$ be absolutely continuous for (F)-effectiveness. On the other hand, the reciprocity relations between $g(u)$ and $G(u)$ show that it is necessary that $q(u)$ be equivalent to a continuous function. It follows, in particular, that the (E_p)-means of K. Knopp [24] cannot be (F)-effective since the corresponding function $q(u)$ is a step function. A direct proof of this fact was first given by C. N. Moore [32].

The question of (L)-effectiveness offers additional difficulties. Suppose, however, that (35) holds, and that there exists a positive, monotone decreasing, continuous function $\mathfrak{C}(u)$ such that

$$(38) \qquad | C(u) | \leqq \mathfrak{C}(u), \quad \int_1^{\infty} \mathfrak{C}(u) du < \infty .$$

Then the corresponding definition $[H, q(u)]$ is (L)-effective, and it will sum the derived series of a function of bounded variation almost everywhere. If a similar inequality is satisfied by the Fourier sine transform of $q(u)-u$, the definition will sum the conjugate series and the conjugate derived series to their proper sums almost everywhere. (A slightly different criterion occurs in [20, third and fourth notes]. See also S. Verblunsky [50].) A definition with all these properties is obtained, in particular, if $Q(u)$ is absolutely continuous and (suitably extended outside of the interval (0, 1)) satisfies (37).

9. *Means Based upon two Sequences*. T. H. Gronwall [14] has recently considered means (f, g) based upon two analytic functions $f(w)$ and $g(w)$. Here $f(w)$ is holomorphic for $|w| \leqq 1$, $w \neq 1$, and $z = f(w)$ gives a one-to-one map of $|w| < 1$ on a domain D in $|z| < 1$ so that $z = 0$ or 1 for $w = 0$ or 1 respectively. The inverse function is supposed to be holomorphic on the boundary of D except at $z = 1$ where

$$(39) \quad 1 - w = (1 - z)^{\lambda} \{ a + (1 - z) \mathfrak{P}(1 - z) \}, \; \lambda \geqq 1, \; a > 0.$$

Further

$$(40)\qquad g(w) = \sum_{n=0}^{\infty} b_n w^n,\ b_n \neq 0 \text{ for every } n,$$

$$(41)\qquad g(w) = (1 - w)^{-\alpha} + \gamma(w),\ \alpha > 0,$$

where $\gamma(w)$ is holomorphic for $|w| \leqq 1$, and $g(w) \neq 0$ for $|w| < 1$.

Gronwall then associates with the series $\sum_0^\infty u_\nu$ the sequence $\{U_n\}$ defined by the formal identity

$$(42)\qquad \sum_{\nu=0}^{\infty} u_\nu z^\nu = [g(w)]^{-1} \sum_{n=0}^{\infty} b_n U_n w^n,$$

and says that *the series is summable* (f, g) *to the sum* s *if*

$$(43)\qquad \lim_{n\to\infty} U_n = s.$$

All definitions (f, g) corresponding to a $\lambda > 1$ are (L)-effective (with corresponding properties with respect to the associated series) since every series summable (C, α) is also summable (f, g) to the same sum if $\lambda > 1$. The restriction $\lambda > 1$ is essential since, e.g., the (E_p)-means of K. Knopp, already mentioned, are (f, g)-means with $\lambda = 1$, and are known not to be (F)-effective. The particular choice

$$(44)\qquad w = 4z(1 + z)^{-2},\quad g(w) = (1 - w)^{-1/2}$$

with $\lambda = 2$ gives the means of de la Vallée Poussin [49] who proved that this definition is (F)-effective. Generalizations of the means of de la Vallée Poussin are also included among the (f, g)-means.

The (VP)-means are but rarely found as special cases of more general definitions. In addition to Gronwall's (f, g)-means, I have to mention some (F)-effective definitions studied by H. W. Bailey [2] which contain the (VP)-means as special cases.

10. *Means of the Closed Cycle.* I. We return to definitions of summation based on a single generating function which we denote now by $K(s)$. *We suppose* $K(s)$ *to be defined for* $s \geqq 0$ *and to be of bounded variation in* $(0, +\infty)$. *Further,* $K(s)$ *shall be continuous to the right at* $s = 0$ *and* $K(0) = 1$. The $K(s)$-sum of the series $\sum_0^\infty u_n$ is then

$$(45)\qquad \lim_{\omega\to\infty} \sum_{n=0}^{\infty} K\left(\frac{n}{\omega}\right) u_n$$

if the limit exists. The conditions stated are necessary and sufficient for the regularity of this definition. In this paragraph we restrict ourselves to the case in which $K(s)\equiv 0$ for $s\geqq 1$, so that (45) is replaced by

$$\lim_{\omega\to\infty}\sum_{n<\omega} K\left(\frac{n}{\omega}\right)u_n, \tag{46}$$

and the general case is taken up in §11 of Part II.

Special cases of such definitions occur very early in the history of the theory of summability. The discussion of the general case goes back to Fejér [11] and H. Weyl [52]. Fejér proved that such a definition includes $(C, 1)$ if both $s^{2+\rho}|K(s)|$ and $s^{2+\rho}|K''(s)|$ are bounded for $s>1$ and a fixed $\rho>0$. Weyl assumed $K(s)$ to be monotone decreasing to the limit zero. In a different form and applied to Fourier series only, we find a closely related definition in the writings of W. H. Young [55, 56] who works with the formula

$$\frac{1}{2}\int_0^\infty [f(x+kt)+f(x-kt)]U_k(t)dt \\ = \sum_{n=-\infty}^{+\infty} f_n e^{nix}\int_0^\infty U_k(t)\cos nkt\, dt, \tag{47}$$

where $U_k(t)$ is a function of bounded variation (and absolutely integrable) over $(0, +\infty)$ which vanishes at infinity, and k is supposed to tend to zero. Young limits his actual discussion to the case in which $U_k(t)$ is independent of k, and gives details only for some special cases.

We return to (46) and put

$$C(t) = \int_0^1 K(s)\cos st\, ds. \tag{48}$$

A necessary and sufficient condition for the (F)-effectiveness of this definition of summability is

$$\int_0^\infty |C(t)|\, dt < \infty. \tag{49}$$

The sufficiency is found, in kernel at least, in Young's results. Young of course starts with $C(t)$ and works backwards to $K(s)$.

It is of interest to note that the Fourier transforms play a role already in Weyl's investigation of Gibbs' phenomenon.

Condition (49) is of the same nature as (34) of §8, and what we have said there regarding the construction of (F)-effective kernels applies, mutatis mutandis, to the present situation. Similarly, the condition (38) is sufficient also for the (L)-effectiveness of the $K(s)$-definitions and for the summability almost everywhere of the derived series. The conjugate series involves similar considerations where, however, the cosine transform has to be replaced by the sine transform.

Some special cases require separate mention. In each case it is understood that $K(s)\equiv 0$ for $s>1$. The choice

$$K(s) = (1 - s^p)^\kappa \tag{50}$$

has been considered by W. H. Young ([55, 56]; details only for $p=1$ and 2). The case $p=1$ of course gives the kernel of the Riesz-Cesàro means. The kernel

$$K(s) = (\log C)^\beta (1 - s)^\alpha \left\{\log \frac{C}{1 - s}\right\}^{-\beta} \tag{51}$$

has been investigated by Bosanquet and Linfoot [4, 5] who have also considered other kernels on the logarithmic scale. They have devoted their attention especially to the relations holding between summability by such kernels and continuity in the mean defined by kernels of the same type. As a further example we may list the kernel

$$K(s) = \left[\cos \tfrac{1}{2}p\pi s\right]^\kappa \tag{52}$$

where p and κ are positive integers of which p is odd. This kernel has been introduced by W. Rogosinski [45, 46] in his investigations on section couplings (Abschnittskoppelungen). The corresponding transformation, which is (L)-effective, can be utilized for a number of different purposes in the theory of Fourier series. The case $\kappa=1$ leads to particularly simple formulas in as much as the generalized sum of the Fourier series is then taken to be

$$\tfrac{1}{2} \lim_{n\to\infty} \left\{ s_n\left(x + \frac{p\pi}{2n}\right) + s_n\left(x - \frac{p\pi}{2n}\right)\right\}. \tag{53}$$

Rogosinski has also investigated other kernels of this general class (even for arbitrary types λ in the sense of M. Riesz), and has proved relative inclusion theorems between his means and those of Riesz.

II. Definitions with Infinite Reference

Of the great number of definitions with infinite reference we can only treat three different classes each of which is of some importance in various branches of analysis. These classes will be the means of the closed cycle, already mentioned in §10, and two types of means used in the theory of analytic continuation.

11. *Means of the Closed Cycle.* II. We return to formula (45). A study of the effectiveness of these definitions with respect to Fourier series has been undertaken by Hille and Tamarkin [unpublished]. If we form the series

$$\sum_{n=-\infty}^{+\infty} K\left(\frac{|n|}{\omega}\right) f_n e^{nix} \tag{54}$$

without assuming that $K(s)$ vanishes for large values of s, it may happen that it does not converge for any values of x and ω for a suitably chosen function $f(x) \subset L$. We can guard ourselves against this by assuming, e.g., that $K(s)$ is monotone decreasing and

$$\int_1^\infty K(s)s^{-1}ds < \infty . \tag{55}$$

This condition ensures the convergence of (54) for $0<\omega<\infty$ whenever $s_n(x) = O(\log n)$, that is, almost everywhere.

But we may also occupy a different point of view (see [22]). We may simply require that whenever $f(x) \subset L$, the series in (54) shall be the Fourier series of a function $f(x, \omega)$ such that

$$f(x, \omega) \to f(x) \tag{56}$$

in some set E_f. A necessary and sufficient condition for (54) to be a Fourier series is that

$$\sum_{n=-\infty}^{+\infty}{}' \frac{1}{n} K\left(\frac{|n|}{\omega}\right) e^{nix} \tag{57}$$

should be the Fourier series of a function of bounded variation (for this range of ideas see M. Riesz [44] where references to the earlier literature are to be found). This condition will be satisfied if we suppose that

$$(58) \qquad \int_0^\infty |C(t)| \, dt < \infty, \quad C(t) = \lim_{a \to +\infty} \int_0^a K(s) \cos st \, ds.$$

Further, if (58) holds, (56) is valid at all points of continuity of $f(x)$, so that the $K(s)$-definition of summation is (F)-effective in this generalized sense, and if (55) is also satisfied it will be (F)-effective in the previous, narrower sense. If there exists a positive, continuous, monotone decreasing function $\mathfrak{C}(t)$ satisfying (38) (of course with $C(t)$ defined by (58)), the definition will also be (L)-effective in the sense that (56) holds in the set where (8) is valid.

The relations between the various means of the closed cycle (Abelian and Tauberian theorems) have recently been elucidated in a brilliant manner by N. Wiener [53]; they are largely governed by the properties of the Fourier transforms of the corresponding kernels. It is likely that the methods of Wiener may be used to great advantage in a study of the effectiveness problems of these definitions. A similar remark would seem to apply to the momental means, the kernels of which appear to be "almost of the closed cycle" in Wiener's terminology.

The so-called "gestrahlte Matrizen" of R. Schmidt [47] define transformations having kernels which usually seem to be "almost of the closed cycle" when not actually "of the closed cycle." The solution of the (F)-effectiveness problem for this class of transformations is likely to be found by Fourier analysis of the kernel.

The means of the closed cycle contain a large number of important special cases. Thus if we choose for $K(s)$ a monotone convex function which tends to zero as $s \to \infty$ and such that

$$(59) \qquad \int_0^1 [1 - K(s)] s^{-1} ds < \infty,$$

we always get an (L)-effective definition (in the narrow sense if (55) or similar condition is satisfied). As classical examples of such a choice we may note the kernels e^{-s} and e^{-s^2} which are associated with the names Abel, Poisson and Fourier, Poisson, Weierstrass respectively, and which have played an important rôle in the history and development of the theory of Fourier series [1, 12, 38, 51].

The kernel of Riemann [40]

$$s^{-2} \sin^2 s \tag{60}$$

is not of such a simple nature, but is nevertheless (L)-effective. A number of variations and modifications of the Riemann kernel have occurred in the literature [9, 11, 39, 42]. We note especially

$$s^{-p} \sin^p s, \tag{61}$$

$$(-1)^m (2m)!\, s^{-2m} \left\{ \cos s - \sum_{k=0}^{m-1} (-1)^k \frac{s^{2k}}{(2k)!} \right\}, \tag{62}$$

$$(-1)^m (2m+1)!\, s^{-2m-1} \left\{ \sin s - \sum_{k=0}^{m-1} (-1)^k \frac{s^{2k+1}}{(2k+1)!} \right\}, \tag{63}$$

where m and p are integers $\geqq 1$, which definitions are all (L)-effective. The extreme case $p=1$ in (61) is particularly interesting. This kernel is not of bounded variation in $(0, +\infty)$, and it is known that the corresponding transformation is not regular. Nevertheless, it was shown to be (L)-effective by Lebesgue [27], and we may notice that condition (58) is satisfied.

Definition (45) is a special case of

$$\lim_{\omega \to \infty} \sum_{n=0}^{\infty} K\left(\frac{\lambda_n}{\omega}\right) u_n \quad (0 \leqq \lambda_n < \lambda_{n+1},\ \lambda_n \to +\infty) \tag{64}$$

which has been proposed by Perron [37]. In some cases it would seem possible to apply a similar analysis to the (F)-effectiveness problem of such means.

12. *Summation by Analytic Convergence Factors.* The problem of analytic continuation of power series has led to a number of definitions of summation. Among these we may distinguish various definitions employing convergence factors $\gamma(n, \alpha)$ which are holomorphic functions of n as well as of α and α is supposed to tend to zero. One possibility of getting such factors is to put $\gamma(n, \alpha) = K(\alpha, n)$ where $K(z)$ satisfies the conditions of regularity stated in §§10 and 11, and, in addition, is holomorphic in

a domain of the complex z-plane containing the positive real axis including the origin. In addition to the choices e^{-z} and e^{-z^2} already mentioned, we may list the kernels

$$\frac{1}{\Gamma(1+z)}, \quad \frac{\cos \pi z}{\Gamma(1+z)}, \quad \frac{\sin \pi z}{\Gamma(1+z)}, \quad \frac{e^{+\pi i z}}{\Gamma(1+z)}, \tag{65}$$

of which the first one has been used by G. Mittag-Leffler [31] and the others by H. von Koch [25]; all these definitions are (L)-effective.

Definitions of this nature, but based upon a more general choice of $\gamma(z, \alpha)$, have been proposed by E. Lindelöf [30]. These definitions embrace as particular cases those of von Koch and Mittag-Leffler quoted above, as well as the definition of Le Roy [29] and one by Lindelöf himself corresponding to

$$\frac{\Gamma(1+(1-\alpha)z)}{\Gamma(1+z)}, \quad \exp\left\{-\alpha(z+1)\log(z+1)\right\} \tag{66}$$

respectively. Both these definitions are (L)-effective since any series summable $(C, 1)$ is summable to the same sum by either method (H. L. Garabedian [13], and D. S. Morse [33], respectively). As a further special case we may mention a definition due to Perron [37], based on the convergence factors

$$\frac{\omega^{z+1}\Gamma(\omega)}{\Gamma(\omega+z+1)}, \quad \omega = \alpha^{-1}, \tag{67}$$

which is easily shown to be (L)-effective.

The Dirichlet series definitions studied by H. L. Garabedian [13] and W. H. Durfee [10] with

$$\gamma(n, \alpha) = e^{-\alpha\lambda_n}, \; 0 \leqq \lambda_n < \lambda_{n+1}, \; \lambda_n \to +\infty, \tag{68}$$

are special cases of Lindelöf's general definition if λ_n is a sufficiently simple analytic function of n. The inclusion relations holding between these definitions and the typical means of M. Riesz show that they are ordinarily (L)-effective unless the sequence $\{\lambda_n\}$ grows either too fast or too slowly.

13. *Summation by Entire Functions.* Another type of means which has been utilized in order to effect analytical continuation of power series is based upon the use of summatory functions,

especially entire functions. We take as the generalized limit of the sequence $\{s_n\}$ the expression

$$(69) \qquad \lim_{\omega\to\infty} [E(\omega)]^{-1} \sum_{n=0}^{\infty} s_n c_{n+1}\omega^{n+1},$$

where

$$(70) \qquad E(z) = \sum_{n=0}^{\infty} c_n z^n, \; c_0 = 1, \; c_n \geqq 0,$$

is an entire function. Such definitions have been proposed by Borel, Lindelöf, Mittag-Leffler, and others (see A. Buhl [6]). It seems that these definitions are ordinarily not even (F)-effective. This was shown by C. N. Moore [32] in the case of Borel's own definition corresponding to $E(z)=e^z$. It is also true of the more powerful definitions based on Mittag-Leffler's function $E_\alpha(z)$, and rough estimates indicate that the same negative result is valid for a wide class of definitions such that the coefficients c_n decrease to zero in a sufficiently regular manner [Hille and Tamarkin, unpublished]. The two classes of definitions in §§12 and 13 have about the same capacity of handling power series, but their effectiveness with respect to Fourier series appears to be entirely different.

III. Definitions for Summation of Integrals

14. *Summable Integrals*. The nth partial sum of a Fourier series is given by formula (10), i.e. by a definite integral involving a parameter n. This suggests a possible application of the theory of summable integrals to our problem. I shall restrict myself to one single illustration, viz. to a definition due to F. Nevanlinna [35] which has close relations to the means of the closed cycle. Let $K(u)>0$ and

$$(71) \qquad \int_0^1 K(u)\,du = 1.$$

Putting

$$(72) \qquad F_K(\omega) = \int_0^1 K(u)F(\omega u)\,du,$$

we can affirm that

$$\lim_{\omega\to\infty} F_K(\omega) = \lim_{\omega\to\infty} F(\omega), \tag{73}$$

whenever the right-hand side exists and is finite. F. Nevanlinna replaces the integral of Dirichlet by the integral of Fourier over a finite interval

$$F(\omega;\, x) = \frac{1}{\pi}\int_a^b f(t)\,\frac{\sin\omega(t-x)}{t-x}\,dt, \tag{74}$$

which has similar convergence and summability properties. He takes this as the function $F(\omega)$ and supposes, in addition, that $K(u)$ is monotone increasing in $0\leqq u\leqq 1$, and that

$$\lim_{\epsilon\to 0}\int_0^{1-\epsilon} K(u)\log\frac{1}{1-u}\,du < \infty\,. \tag{75}$$

He then shows that

$$\frac{1}{\pi}\int_0^1 K(u)\int_a^b f(t)\,\frac{\sin\omega u(t-x)}{t-x}\,dt\,du \to f(x) \tag{76}$$

at every point of continuity. It follows that these definitions of summability are (F)-effective. A. F. Moursund [34] has recently shown that they are also (L)-effective.

15. *Conclusion.* The preceding survey is rather incomplete. In restricting the attention to the (F)- and (L)-effectiveness problems, I had to disregard a number of important developments. The reader will undoubtedly miss references to the theory of multiple series, conjugate series and derived series, to various types of summability, to convergence in the mean, strong convergence and various other forms of generalized convergence, to Tauberian theorems, and to a number of other topics. In spite of having obviously given too little, I may also with due cause be taken to task for having given too much. But I shall be happy if I have at least convinced the reader that the regular definitions of summability which are effective with respect to Fourier series form a vast and important class, and that there is some hope of our being able to bring some order into the bewildering chaos which reigns in this domain at present.

16. *Bibliography.* The subsequent list of references contains only papers referred to in this report, and no pretence is made of its being in any way complete.

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