ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

119. Dr. Leo Zippin (National Research Fellow): *On the Rutt-Nöbeling theorem.*

A new and thoroughly independent proof is given that a locally compact continuous curve C, containing two points x and y such that no N points of C separate x and y, contains at least $N+1$ independent arcs xy. The proof is preferred to that recently published by Nöbeling (in vol. 18 of the Fundamenta Mathematicae, of which advance reprints have just been received) because (1) it is thoroughly inductive, although it has been commonly held that no inductive proof is here possible, (2) it is independent of the Menger "n-Bein Satz" to which the Nöbeling argument is reduced, and therefore gives a new proof of this theorem also, (3) it extends the field of the theorem to locally compact spaces, (4) the proof is much simpler certainly than the combined Menger-Nöbeling papers. The extension to locally compact spaces has this peculiar interest that it strikingly facilitates an induction, which in compact spaces might not be suspected. (Received March 17, 1932.)

120. Dr. Oscar Zariski: *A topological theorem on algebroid singularities.*

The intersection of an algebroid singularity, given by its Puiseux expansion $y=y(x)$, with the boundary of the 4-cell $|x|<$const., $|y|<$const., is stereographically projected into a knot of ordinary space. These knots have been described by K. Brauner, who also gave the generating relations of their fundamental group. That two distinct singularities give rise to distinct knots has been proved by O. Schreier for the particular case in which both knots lie on a torus (singularities of genus 1). The purpose of the present paper is to prove the following general theorem: *If two algebroid singularities are distinct from the algebro-geometric point of view, i.e., if the characteristic numbers which occur in their respective Puiseux expansions are distinct, then they are also topologically distinct, and their fundamental groups are not isomorphic.* (Received March 7, 1932.)

121. Mr. F. G. Dressel: *A generalized boundary value problem for the heat equation.*

Taking the fundamental region as a rectangle, let G_1, G_2 be functions of limited variation along the respective vertical sides of the rectangle, vanishing at the lower vertices and having the property that $G_i(t) = G_i(t-0)$. We consider the function $\phi_1(x, t) + \phi_2(x, t)$ of the type mentioned in the paper by Dressel and Miles presented at the same meeting of this Society (see abstract 123, below). The integrals ϕ_i are extended along vertical segments which on being allowed to approach the sides of the rectangle lead to a system of Stieltjes integral equations. The solution of this system is a unique function of the class considered, having the desired properties of vanishing along the base of the rectangle and taking on the respective values G_1, G_2 along the vertical sides. The usual treatment is the special case in which the boundary values are the integrals of continuous functions. The problem of the more general plane region is being studied. (Received March 7, 1932.)

122. Professor Antoni Zygmund: *On lacunary trigonometric series.*

A lacunary trigonometric series is defined by (1) $\Sigma(a_k \cos n_k x + b_k \sin n_k x)$, $n_{k+1}/n_k > q > 1$. It is proved that if the partial sums of (1) are uniformly bounded below or above on a set E of positive measure, then $\Sigma(a_k^2 + b_k^2)$ converges. The infinite products $\Pi(1 + a_k \cos n_k x)$ are used systematically in this paper for constructing Fourier-Stieltjes series which converge to zero almost everywhere. (Received March 7, 1932.)

123. Mr. F. G. Dressel and Professor E. R. C. Miles: *A class of solutions of the heat equation.*

Consider the Poisson-Stieltjes integral for the heat equation, $u(x, y) = (1/(2\cdot\pi^{1/2}))\int_a^b (1/(y-\eta)^{1/2}) e^{-\mu(\xi)} dF(\xi)$, where $\mu(\xi) = (x-\xi)^2/(4(y-\eta))$ and $F(\xi)$ is a function of limited variation with regular discontinuities. Under slight restrictions as to the manner in which the point (x, y) approaches an interior point x_0 of the interval (a, b) on the characteristic of $y = \eta$, $u(x, y)$ is shown to take on the value $F'(x_0)$ whenever this exists. If $F(t, y)$ denotes the integral from a to t of $u(x, y)$ with respect to x, then $\lim_{y \to \eta} F(t, y) = F(t) - F(a)$. The function $v(x, y) = (1/(2\cdot\pi^{1/2}))\int_a^y (x - x_0)(y-\eta)^{-3/2} \cdot e^{-\mu(x_0)} dF(\eta)$ is analogous to the potential of a double layer, and here it is shown that, according as x approaches x_0 from the right or from the left, $v(x, y)$ will take on the value plus or minus $F'(y-0)$, when this one-sided derivative exists. The function $\phi(x, t)$ defined by integrating $v(x, y)$ from a to t with respect to y has the property that $\lim_{x \to x_0} \phi(x, t)$ is plus $[F(t-0) - F(a)]$ for $x > x_0$ and minus this same value if $x < x_0$. (Received March 7, 1932.)

124. Dr. R. P. Agnew (National Research Fellow): *On summability of double sequences.*

Let $\|a_{mi}\|$ and $\|b_{nj}\|$ be triangular matrices satisfying the conditions (1) for each i, $\lim_{m\to\infty} a_{mi} = 0$; for each j, $\lim_{n\to\infty} b_{nj} = 0$; (2) for each m, $\sum_{i=0}^{m} |a_{mi}| < K$; for each n, $\sum_{j=0}^{n} |b_{nj}| < K$, where K is a constant; and (3) $\lim_{m,n\to\infty} \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mi} b_{nj} = 1$. Corresponding to each double sequence s_{ij}, let $S_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mi} b_{nj} s_{ij}$. We show that if s_{ij} converges to s and if each sufficiently advanced row and column of its transform S_{mn} is bounded, then S_{mn} converges to

s. We show also that if the matrices $\|a_{mi}\|$, $\|b_{nj}\|$ satisfy a certain condition in addition to (1), (2), and (3), and if s_{ij} converges and each sufficiently advanced row and column of S_{mn} is bounded, then each row and column of S_{mn} is bounded. Applications are given. (Received March 25, 1932.)

125. Professor W. A. Hurwitz: *Definition of a field by four postulates.*

The small number of postulates used for the definition of a group in this paper is obtained by means similar to those previously employed by the author. The associative and commutative laws of addition are expressed by one formula; the associative and commutative laws of multiplication and the distributive law, by another formula. Complete independence is verified, even when the set is amplified by postulation of number of elements. (Received March 25, 1932.)

126. Mr. Walter Prenowitz: *Characterization of collineations on the real projective plane.* Preliminary report.

In this paper the following theorem is proved: Any one-to-one point transformation on the real projective plane, which carries the lines of four pencils with non-collinear vertices into lines, is a collineation. (Received March 23, 1932.)

127. Professor I. A. Barnett: *Functional invariants of integro-differential equations.*

The object of this investigation is the study of functional invariants of systems of linear homogeneous integro-differential equations under the Fredholm group. In this paper the single equation

$$y_{tt}(x,t) + \int_0^1 L\,(x,\eta,t)\,y_t\,(\eta,t)d\eta + \int_0 M\,(x,\eta,t)y(\eta,t)d\eta = 0$$

is considered and is subjected to the transformation

$$y(x) = z(x) + \int_0 K\,(x,\eta)\,z\,(\eta)\,d\eta.$$

The induced functional group on the coefficient kernels and their derivatives is then found both in the finite and infinitesimal forms. From this group are deduced the functional equations which the invariants satisfy. A number of functional invariants may be derived by writing the integro-differential equation in a canonical form in which the term involving the first derivative is missing. (Received March 17, 1932.)

128. Mr. Gordon Fuller: *On the invariant character of a set of differential equations.*

This paper deals with a set of equations (the subscripts denote differentiation): $\delta_x+\gamma_y+\alpha_z-\alpha/2=0$, $\gamma_x-\delta_y+\beta_z-\beta/2=0$, $\beta_x+\alpha_y-\gamma_z-\gamma/2=0$, $\alpha_x-\beta_y-\delta_z-\delta/2=0$, which belong to a type that has recently gained prominence in physics. Equations of this type can be proved to be invariant under transformations of coordinates, although they lack symmetry with respect to x, y, and z. The system is treated by introducing a set of four new dependent variables, three of which may be considered as components of a vector. The fourth, which does not transform in any orthodox way, is eliminated by differentiation. The non-linear set of second order equations thus obtained is then written in a form in which the invariance is easily recognizable. A transformation on a vector with components u, v, w involving their partials with respect to x, y, z is used. (Received March 17, 1932.)

129. Professor Tibor Radó: *On approximately conformal maps of surfaces.*

Given a surface $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, such that $EG-F^2>0$, it is possible to find a conformal map of the surface, that is to say a map such that $E=G$, $F=0$. If $EG-F^2$ has zeros, then a conformal map does not generally exist. In his work on the problem of Plateau, the author used *approximately conformal maps*, that is to say maps for which $\iint(E^{1/2}-G^{1/2})^2$ and $\iint|F|$ are arbitrarily small. The object of this paper is to develop an existence theorem for such maps and applications of this theorem to the problem of Plateau. (Received March 17, 1932.)

130. Mr. D. S. Nathan: *Sphere geometry and the conformal group in function space.*

This paper develops the sphere geometry of euclidean function space R_x as the analogue of the elementary sphere geometry of euclidean n-space. The element is the sphere in R_x, namely,

$$\sigma \int_0^1 f^2(y)\, dy - 2\int_0^1 \phi(y) f(y)\, dy + \omega = 0,$$

where $f(x)$ is a continuous function on the interval $0 \leqq x \leqq 1$ representing the variable point, $\phi(x)$ is a given continuous function on this interval, and σ, ω are given real numbers on the interval $0 \leqq \sigma$, $\omega \leqq 1$. We call $\phi(x)$, σ, ω the homogeneous sphere coordinates. Conformal transformations are defined as those linear functional transformations in $\phi(x)$, σ, ω which take spheres of zero radius into spheres of zero radius. They also preserve orthogonality of spheres. A conformal transformation whose Fredholm determinant is not zero is called non-singular. The totality of non-singular conformal transformations forms a group in the sense that the succession of two transformations gives rise to another of the same kind. The infinitesimal conformal transformations are found to be identical with the "regular infinitesimal conformal transformations in homogeneous coordinates" which Kowalewski obtained as the angle-preserving transformations in R_x. The one-parameter group of non-singular conformal transformations generated by an infinitesimal conformal transformation is determined. (Received March 17, 1932.)

131. Dr. A. T. Craig: *Some properties of normally correlated variables.*

Let $t_1, t_2, \cdots, t_n$ be n variables each subject to a Gaussian law of probability. Let t_i, $(i=1, 2, \cdots, n-1)$, be normally correlated with t_{i+1} with the coefficient of correlation $r=r_i$. No assumptions are made regarding the correlation between the variables other than that of the correlation between them in adjacent pairs. In the present paper, a set of functions $\psi_{ij}(t_i, t_j)$, $i+1<j\leqq n$, is determined such that $\psi_{ij}(t_i, t_j)$ is the correlation function of the characters t_i and t_j wherein t_i is possessed by one individual and t_j is possessed by a second individual, the two individuals having been paired identically as to the character t_{j-1}. A correlation coefficient $r=r_i \cdots r_{j-1}$ measures the correlation between the two variables when paired in this manner. (Received March 18, 1932.)

132. Dr. Selby Robinson: *Certain cardinal numbers associated with a point in a general topological space.*

This is a study of equalities and inequalities involving certain numbers associated with a point p. One of these numbers is the power of the least complete family of neighborhoods of p, and another is the power of the least family of neighborhoods whose product is not a neighborhood of p. The equality of these two cardinals implies the equality of all the cardinals considered, and is a necessary and sufficient condition that p have a monotone complete family of neighborhoods. An equivalence which was previously proved by E. W. Chittenden and the author (this Bulletin, vol. 37, p. 628) for spaces V_ω, is extended in a modified form to spaces in which every point has a monotone complete family of neighborhoods. (Received March 18, 1932.)

133. Miss I. M. Schottenfels: *On the interchange of limit and integral for classes of functions in real, complex, and quaternionic numbers.*

This paper develops theorems for the general case of the interchange of limit and integral for classes of functions of real, complex and quaternionic numbers, and incidentally shows that the theorems in the papers of R. L. Jeffery and T. H. Hildebrandt on this subject are special instances of the theorems in this paper. In conclusion it is shown that limits and the Lebesgue, Riemann, and Riesz integrals are interchangeable, and the identity of these integrals is established, as was recently mentioned in a paper by W. M. Whyburn. (Received March 9, 1932.)

134. Professor G. A. Bliss and Dr. M. R. Hestenes: *Sufficient conditions for a problem of Mayer in the calculus of variations.*

In this paper sufficient conditions for a minimum are deduced for a modification of the classical problem of Mayer in the calculus of variations. The problem is that of finding in the class of arcs $y_i=y_i(x)$ $(i=1, \cdots, n; x_1\leqq x\leqq x_2)$ satisfying the differential equations and end conditions $\phi_\alpha(x, y, y')=0$, $(\alpha=1, \cdots, m)$, $\psi_\rho[x_1, y(x_1), x_2, y(x_2)]=0$ $(\rho=1, \cdots, 2n+1)$ one which minimizes a function $g\ [x_1, y(x_1), x_2, y(x_2)]$. The classical problem of Mayer is the special case for which the function g to be minimized is the end value $y_1(x_2)$ and the conditions $\psi_\rho=0$ specify the other end values of the variables $x, y_1, \cdots, y_n$. This problem is equivalent to one formulated by Bolza for which sufficiency theorems have been deduced by Morse and Bliss. In both cases, however, the proofs depend upon assumptions which prevent the problem of Bolza from being equivalent to the most general problem in the Mayer form. One of the most important features of this paper is the use of an $(n+1)$-dimensional field in $(x, y_1, \cdots, y_n)$-space rather than the n-dimensional fields which have been used hitherto by Kneser, Egorov, and Larew for the classical problem of Mayer and which do not seem to be applicable here. It is interesting to note that the auxiliary minimum problem associated with the second variation is abnormal. (Received April 8, 1932.)

135. Dr. M. R. Hestenes: *Sufficient conditions for a problem of Mayer with variable end-points.*

The problem considered in this paper is that of finding in the class of arcs $y_i = y_i(x)$, $(i = 1, \cdots, n;\ x_1 \leqq x \leqq x_2)$, satisfying the differential equations and end-conditions $\phi_\alpha(x, y, y') = 0$ $(\alpha = 1, \cdots, m < n)$, $\psi_\rho[x_1, y(x_1), x_2, y(x_2)] = 0$ $(\rho = 1, \cdots, p < 2n+1)$ one which minimizes a function $g[x_1, y(x_1), x_2, y(x_2)]$. This is the problem of Mayer with variable end-points as formulated by Bliss (Transactions of this Society, vol. 19 (1918), p. 305). Sufficient conditions by Bliss and Hestenes for the case when $p = 2n+1$ are applied to establish similar conditions in the general case. The proof for the case when $p < 2n+1$ is different from that for the case $p = 2n+1$, but makes use of the sufficiency theorem for the latter case. The method used is related to that used by Bliss for the problem of Bolza (Annals of Mathematics, (2), vol. 33 (1932)). Theorems on normality are given, and the necessary condition of Mayer for variable end-point problems is given a new form. The results obtained are equally applicable to the problem of Bolza considered as a problem of Mayer with variable end-points but the proofs hitherto given by Morse and Bliss for the problem of Bolza do not apply to the general case of the problem of Mayer. (Received April 8, 1932.)

136. Professor W. H. Durfee: *Summation factors for double series.*

The double series considered in this paper are of the type $J(z, w) = \sum_{m,n=1}^{\infty} a_{m,n} z^{f(m)} w^{g(n)}$, where $\sum_{m,n=1}^{\infty} a_{m,n}$ is summable by some Cesàro mean with the value s and has bounded partial sums, z and w are complex variables, and $f(m)$ and $g(n)$ are logarithmico-exponential functions. Sufficient conditions on $f(m)$, $g(n)$ are given so that the series shall converge for all values of z and w within the unit circle, and so that $\lim J(z, w) = s$ as z and w independently approach unity from within the unit circle. (Received March 15, 1932.)

137. Mr. Harry Matison: *On the nature of the solutions of certain types of integro-differential equations.*

Equations of the type $\partial z(x, t)/\partial t = F[z(r, s); x, t]$ are considered. Under suitable hypotheses involving the existence and continuity of the Fréchet differential of the right hand side, the existence of the derivative of the solution with respect to x is established. (Received March 16, 1932.)

138. Mr. R. S. Martin: *Contribution to the theory of analytic functions in abstract spaces.* Paper II.

In a preliminary report under the same title (this Bulletin, Nov., 1931, p. 824, abstract 371) the author announced results on the properties of analytic correspondences in Banach spaces with closure under multiplication by numbers of a ring. The special cases of closure under the fields of real and complex numbers are examined in more detail. The equivalence of various possible definitions of abstract polynomials is shown to depend on whether or not the field contains a complex quantity. Incidentally an elegant treatment of the Fréchet theory of abstract polynomials is included. These results are employed to obtain further development of analytic function theory in abstract spaces. (Received March 17, 1932.)

139. Professor M. A. Basoco: *On the trigonometric developments of certain doubly periodic functions of the second kind.*

In this paper we are primarily concerned with deriving the explicit arithmetized Fourier series expansions for the functions

$$\phi_{\alpha\beta\gamma}(x, y) \equiv \theta_1'^2[\theta_\alpha^2(x+y)/\theta_\beta^2(x)\theta_\gamma^2(y)]$$

where $\theta_\mu(z)$ are the elliptic theta functions of Jacobi, (x, y) are independent complex variables and (α, β, γ) are certain sixteen triads out of the possible sixty-four which can be selected from the numbers 0, 1, 2, 3. These functions belong to the class of doubly periodic functions of the second kind (in the sense of Hermite), and hence can be treated by the methods of Teixeira given in the Journal für Mathematik, vol. 125. A generalization of these results lead to certain formulas of decomposition which when applied to the functions $\phi_{\alpha\beta\gamma}(x, y)$ yield the desired expansions. These developments may be used to obtain formulas of the Liouville type involving arbitrary numerical functions (e.g. see E. T. Bell, Transactions of this Society, vol. 22 (1921); Colloquium series, VII, p. 88, 1927). Finally, the developments obtained can be made to yield the expansions of certain functions of the third kind. (Received March 15, 1932.)

140. Dr. I. J. Schoenberg: *Concerning the theory of systems of linear inequalities.*

This paper contains a geometric exposition of the fundamental results of Minkowski and Farkas with the additions given by Haar, who presented an elegant geometric treatment of the subject (Szeged Acta, vol. 2 (1924), fasc. 1, pp. 1–14) based, however, on a rather restrictive assumption which asks that the set of points in n-space which are solutions of the given system of linear inequalities in n variables has interior points. In the present paper this assumption is not made. Moreover some of the results have been extended to cases which have not been treated before. This paper, together with the paper presented by the author at the New Orleans annual meeting, will appear in the Transactions of this Society. (Received March 15, 1932.)

141. Professor W. A. Manning: *On primitive groups containing circular permutations of degree p^α or $2p^\alpha$.*

If a non-alternating primitive group of degree n contains a circular permutation of prime degree m, then $n \leqq m+2$. This was proved by C. Jordan. In the present paper it is shown that the theorem remains true if $m=p^\alpha$ or $2p^\alpha$ (p prime). (Received March 18, 1932.)

142. Professor J. S. Turner: *Quadratic cycles.*

The quadratic cycles are of the type $d-x_1^2=r_0r_1$, $d-x_2^2=r_1r_2$, $\cdots$, $d-x_n^2=r_nr_0$, where x, r, and d are positive integers, d not a perfect square, obtained when $(\sqrt{d}-x)/r$ is converted into a continued fraction. The results obtained in this paper include (a) necessary and sufficient conditions that the cycle may be symmetric, (b) the number of symmetric cycles for a given d, (c) unless $r_0=r_1$ and $d=2x_1^2$ or $5x_1^2$, the cycle contains an $r<\sqrt{d}/2$. (Received March 7, 1932.)

143. Mr. C. R. Worth: *Subvarieties of a field.*

The postulates for a field given by Dickson (Transactions of this Society, 1905, p. 198) are shown to be completely independent in the sense of E. H. Moore for both finite and infinite classes when closure is assumed for both addition and multiplication. The set obtained by adding the postulate of commutativity of addition is shown to give consistent number systems in almost all cases. Of the 2^{13} number systems arising from a modification of a postulate system due to Huntington (ibid., p. 181) the consistency of all except approximately 190 has been settled. The purpose of this paper is not to show the complete independence of the postulate systems under consideration, but to determine the consistent subvarieties of a field. (Received March 15, 1932.)

144. Professor G. D. Birkhoff and Dr. W. J. Trjitzinsky: *Analytic theory of singular difference equations.*

In this paper the authors develop the analytic theory of linear difference equations when no restrictions whatsoever are placed on the formal series solutions. The fundamental result is as follows. Depending on the exponential factors in the formal series the complete neighborhood of infinity is divided into a finite number of regions, separated by curves with a limiting direction at infinity; if R is such a region, there exists a fundamental set of solutions with elements analytic and asymptotic to the formal series in R. The converse and the related Riemann problems are also solved. The paper is based, on one hand, on the earlier papers of Birkhoff on difference equations; on the other hand, essentially new methods are used involving factorization and "group summation." This work will appear in the Acta Mathematica. (Received April 9, 1932.)

145. Dr. Walter Bartky and Professor R. W. Barnard: *Note on the solution of normal systems of linear differential equations with constant coefficients.*

A fundamental set of solutions of the differential equations $dx_idt = \Sigma_j \kappa_{ij} x_j$, $(i, j = 1, \cdots, n)$, where $\kappa = \|\kappa_{ij}\|$ is a matrix of constants, is $\phi(\kappa) = \Sigma_i e^{\alpha_i t} \cdot [\delta + (\kappa - \alpha_i \delta)t/1! + \cdots + (\kappa - \alpha_i\delta)^{m_i - 1} t^{m_i - 1}/(m_i - 1)!] \Pi_{j \neq i} (\kappa - \alpha_j \delta)^{m_j}$. The α_i are characteristic values for κ of multiplicity m_i and δ is the identity matrix. That $\phi(\kappa)$ is a solution is readily verified by substitution and noting that κ satisfies its characteristic equation. Since the matrix $\phi(\kappa)$ is a polynomial in κ, its characteristic values are $\phi(\alpha_i) = e^{\alpha_i t} \Pi_{j \neq i} (\alpha_i - \alpha_j)^{m_j} \neq 0$ and, therefore, $\phi(\kappa)$ is a fundamental set. (Received April 9, 1932.)

146. Professor Tibor Radó: *On the second variation of the area-integral.*

Schwarz has already constructed examples of minimal surfaces which do not have a smallest area. The object of this paper is to call attention to examples which can be discussed in a perfectly elementary way, and also to point out that certain uniqueness theorems, proved by the author in previous papers, lead to sufficient conditions for an absolute minimum. (Received March 16, 1932.)

147. Professor O. E. Glenn: *Sketch of an important solvable problem in formal modular covariants.* Preliminary report.

The author proved in 1915 that the binary form with arbitrary coefficients,

$$f_m = a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m,$$

is reducible, modulo 2, in terms of covariants of f_m of orders <4, together with the known universal covariants of the total group, (mod 2). It follows that the system of covariants, modulo 2, of any f_m, $m<8$, will be known when the joint system of (f_1, f_2, f_3) has been determined. The object of this investigation is to determine the latter simultaneous system. (Received March 16, 1932.)

148. Professor A. D. Michal and Mr. J. L. Botsford: *The new Einstein geometry and some of its extensions.*

We consider an n-dimensional Riemannian space R_n, to each point of which is associated an m-dimensional linear vector space $V_m (m \geqq n)$. Correspondences between vectors in V_m and vectors in the associated tangent space are defined. Geometries involving a König linear connection and a set of normal orthogonal exceptional vectors are developed. Several theorems on the differential invariants of such geometries are proved. The results, for $m=5$, $n=4$, include those considered recently by Einstein and Mayer (Sitzungsberichte der Preussischen Akademie der Wissenschaften, December, 1931, pp. 541–557). (Received March 17, 1932.)

149. Professor H. A. Simmons: *Further generalizations of certain classes of extreme numbers relative to certain symmetric equations in n reciprocals.*

In two papers which were presented respectively at the Chicago and New Orleans meetings of the Society in 1931, the author identified certain rather general classes of maximum numbers and minimum numbers relative to certain elementary symmetric equations in n reciprocals. In this paper considerable extensions of the previous results are made by dealing with certain symmetric equations that are not elementary symmetric. (Received March 17, 1932.)

150. Professor R. L. Jeffery: *Non-absolutely convergent integrals with respect to a function of bounded variation.*

Let $\alpha(x)$ be a function of bounded variation on (a, b), and $F(x)$ a function with discontinuities of the first kind only. The derivative of F with respect to α, and the determination of F when the derivative is given and is finite at each point is studied. The latter problem leads to a theory of non-absolutely convergent integrals with respect to a function of bounded variation analogous to the Denjoy-Khintchine integral with respect to the variable x. Some of the results are: (1) If the derivative of F with respect to α, $D_\alpha F$, exists at each point it is measurable relative to α. (2) If $D_\alpha F$ is finite at each point, then it is Denjoy-Khintchine integrable with respect to α and its integral is equal to $F(b)-F(a)$. (3) If the function $F(x)$ is the Denjoy-Khintchine integral with respect to α of the function $f(x)$, then the approximate derivative of $F(x)$ with respect to α is equal to $f(x)$ except for at most a set of α-measure zero. (4) A discussion of the properties of Denjoy-Khintchine integrals with respect to α, including a formula for integration by parts, and the second mean value theorem for integrals. (Received April 1, 1932.)

151. Dr. A. E. Currier: *The variable end point problem of the calculus of variations including a generalization of the classical Jacobi conditions.*

Let H and H' be two transverse families of extremals in m-space $(z) = (z_1, \cdots, z_m)$. Let g be an extremal belonging to both H and H'. Let M and M' be transverse manifolds of H and H' respectively. Let V and V' be the Hilbert integrals associated with H and H' respectively. We are concerned with a variable end point problem in which the end points vary on M and M' respectively. We define a new invariant function $I(z, w) = \Sigma_{i,j=1}^{m}[V_{z_iz_j} - V'_{z_iz_j}]w_iw_j$, where the partial derivatives of the Hilbert integrals are evaluated at the point (z) on g, and (w) is arbitrary. Associated with each point (z) of g we obtain a set of necessary conditions for a relative minimum along g and also a set of sufficient conditions. These conditions involve the number of focal points of H on the segment of g between M and (z), the number of focal points of H' on the segment of g between (z) and M', and the type numbers of the quadratic form $I(z, w)$. In case (z) lies outside the segment of g between M and M' our conditions give the generalization for m-dimensional space of the classical Jacobi conditions of the problem in the plane. (Received April 6, 1932.)

152. Professor Clifford Bell: *Alternant surfaces.*

The surfaces obtained from the third order alternant, $A = |\alpha^m\beta^n\gamma^p|$, $m<n<p$, by setting $\beta/\alpha = x$, $\gamma/\alpha = y$ and $A/[(\beta\gamma)^m\alpha^{m+n+p}] = z$, are shown to

fall into four types. Three concurrent lines are common to all the surfaces along which z vanishes, and their intersection is a common minimax point. By means of these surfaces it is possible to determine the existence, or non-existence, of values of α, β, γ, no two of which are equal, such that A vanishes. (Received April 18, 1932.)

153. Professor Clifford Bell: *On some properties of polygons related to the cuspidal cubic and a line.*

The n-sided polygons studied are those having $n-1$ vertices on, and n sides tangent to, the cuspidal cubic, the remaining vertex being on a given line. The polygons, so related to the cubic and a given line, are shown to be perspectively related for certain positions of the line. Other projective properties of the polygons are developed, together with theorems concerned with the lines through their corresponding vertices. (Received April 18, 1932.)

154. Dr. H. B. Curry: *Some properties of equality in combinatory logic.*

This paper deals with properties which are of the same general type as those considered in a previous paper, *Apparent variables from the standpoint of combinatory logic* (see abstract 37-1-60), and which result from the introduction of certain new axioms for equality and implication. The results include certain formulas expressing formally the properties of equality previously established informally. The principal theorem, however, amounts to a justification of what may be called the postulational method of proof, i.e., the method whereby, wishing to establish that a certain conclusion is a formal consequence of certain premises (both premises and conclusion being propositional functions of n variables), one argues as follows: Let $x_1, x_2, \cdots, x_n$ be arbitrary entities such that the premises are satisfied, then for these x's the conclusion is also satisfied. The justification consists in the exhibition of a general method whereby, given any valid argument of the above nature, the same may be transformed into a demonstration, within the formalism of combinatory logic, of a formula whose interpretation is the fact required to be proved. (Received April 21, 1932.)

155. Mr. B. F. Groat: *Misuse of postulations in theory of viscosity.*

Studies in Newtonian similitude (this Bulletin, vol. 25, p. 52; vol. 36, pp. 151-194; Proceedings of the American Society of Civil Engineers, Aug. 1931, pp. 938-989; The Journal of Engineering Education, Jan. 1932, pp. 379-383) disclose misuse of postulation in the beginnings of the kinetic theory of viscosity. Error is first indicated by the axiom: no solution in mathematical physics is valid if an assumption is necessary to it (discloses similar error in hydrodynamics, Proceedings of the American Society of Civil Engineers, p. 959). Solutions result only from existing theory. Assumptions are precluded, or, if of undeniable force, they postulate new theories, not the old. Empiricism may serve in some engineering problems, but not in a formal science based upon "a priori" postulates which agree with experience throughout the region of practical observation. This mathematical philosophy is confirmed by mathematical analysis showing that Maxwell's assumed free-path distribution, creates fallacies appearing in all books treating the subject. When expunged viscosity theory will become valuable. Resulting errors in physical constants also need correction. It is remarkable that Newton and Maxwell each made an error in viscosity,—Newton by misusing his correct postulation; Maxwell by making an empirical postulation. (Received May 4, 1932.)

156. Professor R. D. Carmichael: *Summation of Functions of a Complex Variable.*

The main object of this paper is to develop a theory of the principal solutions of the equations $F(x+\omega)-F(x)=\omega\phi(x)$, $G(x+\omega)+G(x)=2\phi(x)$. In part I a definition of principal sums is employed which is closely related to that of Nörlund (Acta Mathematica, vol. 44 (1922), pp. 71–211) and the theory is developed for the case of functions $\phi(x)$ having a certain defined asymptotic character at infinity. In part II generalized sum formulas of the Euler-Maclaurin type are introduced and are employed in constructing principal sums of functions of exponential type; and these sums are developed in Fourier series by a direct method. In part III necessary and sufficient conditions are given for the expansion of functions in series of Bernoulli-Hurwitz functions, including the special case of Bernoulli polynomials. (Received May 6, 1932.)

157. Dr. Hassler Whitney: *On 2-congruent graphs.*

It is shown that if there is a 1-1 correspondence between the arcs of the graphs G and G' such that if a set of arcs of one graph form a circuit (or a subgraph of nullity 0) then the corresponding arcs of the other do also, then G can be carried into G' by operations of the two following types. (1) The graph is broken at a single vertex into two pieces, or two pieces are joined at a vertex. (2) If the graph is composed of two connected graphs H_1 and H_2 which have just the two vertices a and b in common, then H_1 is turned around so that its arcs which were formerly on a (b) are now on b (a). By definition, G and G' are 2-congruent. (Received May 7, 1932.)

158. Dr. H. L. Garabedian: *Note on a theorem due to Bromwich.*

The theorem of Bromwich under consideration gives sufficient conditions in order that a definition of summability with infinite matrix of reference shall include Cesàro summability of order k. The present paper reestablishes this theorem by a more direct and shorter method than that used by Bromwich. Moreover, this proof affords a method of exhibiting a k-fold summability with infinite matrix of reference, analogous to well known definitions of summability with finite matrices of reference which make use of repeated means, for any convergence factor which satisfies the conditions of the theorem under discussion. (Received May 10, 1932.)