THE N. R. C. REPORT ON ALGEBRAIC GEOMETRY

Selected Topics in Algebraic Geometry. Bulletin of the National Research Council, Number 63. Report of the Committee on Rational Transformations: Virgil Snyder (Chairman), A. B. Coble, Arnold Emch, Solomon Lefschetz, F. R. Sharpe, C. H. Sisam. Published by the National Research Council of the National Academy of Sciences, Washington, D. C., 1928. 395 pp.

The Division of Physical Sciences of the National Research Council in 1925 appointed a committee of six mathematicians, consisting of Virgil Snyder (Chairman), A. B. Coble, Arnold Emch, Solomon Lefschetz, F. R. Sharpe and C. H. Sisam, to draw up a report on the work to date in the field of rational transformations. The committee, after several meetings and a great amount of work both individually and collectively, submitted this report. As stated in the preface, although the main theme is the rational transformation, the committee did not confine itself strictly to this field, but considered broader topics involving or allied to geometric transformations both rational and irrational in the plane, three-space, and hyperspace. This broader treatment makes the report still more valuable.

The report consists of seventeen chapters written individually by the various members of the committee. Each chapter, complete in itself, gives a brief development of its topic, citing references to authors throughout. The references are arranged alphabetically according to the name of the author and are numbered serially. The complete list giving the author and periodical reference for each citation is found at the end of each chapter. The chapters are subdivided into sections and the section in which the reference is cited is given in a bracket at the right of the reference in the list at the end of the chapter for all chapters except 1, 2, 10, 14.

In the report, 2794 papers written by 1306 different authors are cited. References to these papers are skilfully woven into the discussion with concise appraisals of the contribution of the paper to the development of the subject. The report would be of great value if it were merely a bibliography of this size, but the pithy description of the content of the paper cited, together with the cross reference scheme described above enabling one who knows the name of the author to find the description of the paper in the report, increases its value many fold. These brief descriptions come from original sources and are therefore much more dependable than if obtained from synopses.

This comprehensive report is particularly pleasing in that so few essential typographical errors occur. This is chiefly due to the great care taken by the authors, and especially by the Chairman, in preparing the manuscript and reading proof. Practically all of the typographical errors that do occur are minor errors in punctuation, spelling of names and foreign words, size and style of type. The reviewer has not found one that would be really confusing to a careful reader.

In lines 8, 9, page 84, it is stated that Montesano proved (1905) that there exist only four symmetric types of plane Cremona transformations, namely those of order 2, 5, 8, 17. This was previously established by Bertini* in 1877 in a footnote of the page cited. The first line of the table given in the footnote is incorrect. The equations (8), (9), page 252 from which the table is derived are, however, correct. Moreover, the correction in the solution was evidently made by Bertini that same year inasmuch as another table giving the characteristics of these four symmetric types correctly occurs as an erratum at the bottom of the second index page of the same volume.

In addition to being a descriptive bibliography, this report is a series of concise treatises covering the most important phases of algebraic geometry. It is indispensable to all research workers in that field.

Mathematicians are deeply indebted to the Chairman and members of this committee for their unselfish devotion to the preparation of this able report.

A brief description of the contribution of each member of the committee follows.

VIRGIL SNYDER: CHAPTERS 5, 6, 7, 9, 11.

Chapter 5. *Multiple correspondences between two planes.* The properties of (1, 2) point correspondences (or plane involutions of order two) and the birationally independent types of these transformations are derived. The four involution types of Cremona transformations, treated fully in Chapter 4, are briefly described here since the two image points in the simple plane are in (1, 1) correspondence, giving rise to a Cremona involution associated with each involution of order two. Involutions of general order are next discussed, followed by cyclic involutions of general period and (m, n) point correspondences. In this chapter, 127 papers are cited.

Chapter 6. *Involutions on rational curves.* The geometry on a rational curve is treated here only from the standpoint of associated linear involutions. After discussing involutions on the line and conic, a section is devoted to the general properties of an involution of order n without regard to carrier. Following this, involutions on rational plane curves and rational space cubics and quartics are treated and finally involutions on rational carriers in hyperspace. This chapter cites 253 papers.

Chapter 7. *Correspondences on non-rational curves.* Chasles' principle of correspondence for rational curves is discussed as to history, applications, and extensions, notably the extension by Cayley and Brill to non-rational curves. Zeuthen's formula relating the characteristics of a multiple correspondence between two curves is given, followed by applications of correspondences to cubics and quartics and a treatment of the Brill-Noether theory. In the discussion of the reduction of singularities, the author mentions his own paper establishing the existence of all algebraic plane curves with distinct nodes. It should be said that this paper marks the first successful attack on the problem of the existence of algebraic curves with assigned singularities. The chapter closes with a discussion of irrational involutions and the relations of involutions to enumerative geometry. This chapter cites 434 papers.

Chapter 9. *Involutions and* (1, 2) *correspondences in* S_3, S_3'. Beginning with

* *Richerche sulle transformazioni nel piano,* Annali di Matematica, (2), vol. 8 (1877), p. 271.

the simplest case, that of reciprocal radii, involutorial transformations of space are discussed. A full treatment of (1, 2) correspondences is given, followed by involutions of lines and systems of lines associated with (l, m) correspondences. Some interesting extensions and exceptions in four dimensions are given. The chapter cites 160 papers.

Chapter 11. *Multiple correspondence and mapping in space and hyperspace.* The chapter opens with a discussion of particular cases of (l, m) point correspondences in S_3. The mapping of the lines of S_3 satisfying one or no conditions on the points of S_3 or S_4 respectively and of linear line complexes on other elements are treated in some detail. Multiple correspondences in hyperspace are then discussed and finally compound involutions, that is, (m, n) correspondences composed of two involutions of orders m and n such that the two image points in the multiple planes are connected by a Cremona transformation. The chapter cites 186 papers.

ARTHUR B. COBLE: CHAPTERS 4, 8.

Chapter 4. *Planar Cremona transformations.* The first four sections of this chapter deal with properties of Cremona transformations when the two corresponding planes are not necessarily superposed. The general properties of Cremona transformations are developed and these transformations are then classified. Linear transformations associated with Cremona transformations, products and groups of Cremona transformations are discussed together with their geometric interpretations. In sections 5–10, the corresponding planes are superposed. Fixed points and curves and cyclic sets of points are discussed briefly. Involutorial Cremona transformations are next treated quite fully beginning with Bertini's four birationally independent types and proceeding to other plans of classification and methods of construction. Descriptive references are given to many proofs of the fundamental theorem that every Cremona transformation can be expressed as a product of quadratic transformations. Closely associated with this theorem are many theorems, general and particular, concerning the birational equivalence of linear systems of curves. This subject is fully treated. Periodic transformations and groups of transformations are discussed and the chapter closes with some applications of Cremona transformations to geometry and algebra. The chapter contains 320 citations.

Chapter 8. *Cremona transformations in space and hyperspace.* The description of space transformations follows the same order as for the plane. Involutorial space transformations are, however, omitted and are treated in Chapter 9. Regular groups of Cremona transformations in hyperspace are discussed. Also at the end of some of the sections, the results are extended to hyperspace. Many of the properties of Cremona transformations of higher space are similar to those of the plane. The most important different feature for S_3 is that a general Cremona space transformation can not be expressed as a product of a finite number of Cremona space transformations of lower order. The chapter cites 224 papers.

ARNOLD EMCH: CHAPTERS 1, 2, 10.

Chapter 1. *Quadratic Cremona transformations.* Quadratic Cremona transformations are first treated from the historical point of view beginning with the first known examples and concluding with the discovery of the general form of such transformations by Cremona in 1862. The properties of general quadratic transformations are developed synthetically and algebraically, after which follows a discussion of special forms of these transformations, including circular inversion, and geometric applications. Groups of transformations are now treated and finally systems of linear transformations associated with quadratic Cremona transformations. The chapter cites 265 papers.

Chapter 2. *Analysis of singularities of plane algebraic curves.* The historical treatment follows the plan of Enriques in the second volume of *Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche* and deals with the development of the theory of algebraic singularities from many points of view. It also includes a short discussion of the still largely unsolved problem of the existence of algebraic curves with assigned singularities. The analysis of singularities is treated chronologically. The projection of a singular plane curve into a non-singular hyperspace curve is discussed. The chapter closes with an application of the method of resolving a singularity to determining the number of intersections of two curves with given singularities at a given point. In the chapter, 171 papers are cited.

Chapter 10. *Reduction of singularities of space curves and surfaces.* In a very brief chapter, the author treats developments in this subject since 1915, referring the reader to accounts by Castelnuovo, Enriques, and Berzolari for earlier contributions. Seventeen papers written since 1915 are cited. There is also a list of earlier works containing 41 items. The titles of these papers are given but they are not discussed.

SOLOMON LEFSCHETZ: CHAPTERS 15, 16, 17.

Chapter 15. *Transcendental theory.* The chapter opens with a discussion of Riemann surfaces and abelian integrals associated with an algebraic curve. Sections 2–5 deal with a non-singular, irreducible surface F in any space as follows: Linear systems of curves on F and the properties of F associated with them; topological properties of F; generalized abelian integrals associated with F; the distribution of curves on F into discrete continuous sets of linear systems. Extensions of some of the above results to varieties in hyperspace are briefly treated. The chapter concludes with a glossary of terms and notations used in this and the two succeeding chapters. Sixty-two papers are cited. A list of recent books is also given.

Chapter 16. *Singular correspondences between algebraic curves.* Algebraic correspondences between the points of the same or distinct curves are singular if they impose restrictions on the Riemann matrices of the curves, otherwise non-singular. Non-singular correspondences are treated in Chapter 7. Singular correspondences between points on the same curve are developed and discussed from several points of view, and this is followed by a short section on the closely allied theory of correspondences between the points of two distinct curves. The next problem dealt with is that of determining all correspondences between the points of a fixed and a variable curve, and the inverse problem, which latter involves the construction of regular Riemann surfaces. The reduction of abelian integrals and its relation to the determination of irrational involutions is treated briefly. Irrational series of points on a curve and their relations to correspondences between two curves are investigated. The chapter closes with a discussion of birational transformations of points on a non-rational curve. One hundred nine papers are cited.

Chapter 17. *Hyperelliptic surfaces and abelian varieties.* First there is given a condensed but comprehensive review of the basic definitions and properties of multiply periodic and related functions, closing with the definition of an abelian variety. The study of hyperelliptic surfaces is chronological. The subject is fully treated and the important investigations described with clarity and some detail. The general properties of abelian varieties are developed. A treatment of impure matrices and the varieties associated with them follows. The relation of geometric transformations to complex multiplication is then shown, and this is followed by a discussion of complex multiplication. One hundred forty-nine papers are cited.

FRANCIS R. SHARPE: CHAPTERS 3, 12, 13.

Chapter 3. *Linear systems of plane curves.* In three pages, the author gives a comprehensive treatment of linear systems of plane curves including a discussion of superabundant systems, adjoints, and composite systems. Twenty-one papers are cited.

Chapter 12. *The mapping of a rational surface on a plane.* Methods of mapping all rational three-space surfaces of orders less than or equal to five are given and those cases which have been investigated for orders greater than five, including the general order n. The general theory of mapping a rational surface on a plane is developed, and this is followed by discussions of the projection of surfaces of hyperspace into surfaces of S_3, the relation of rational surfaces having plane sections of given genus to linear systems of curves of least order and that genus, rational surfaces that may be mapped on multiple planes, general conditions for a surface to be rational, and some special rational surfaces. One hundred forty-seven citations are made.

Chapter 13. *The mapping of a rational congruence on a plane.* This brief chapter deals with the mapping on a plane of congruences with a finite or a singly infinite number of singular points, closing with a discussion of congruences defined by quadrics and projectivities. Twenty-eight papers are cited.

CHARLES H. SISAM: CHAPTER 14.

Chapter 14. *Involutions on irrational surfaces.* After defining an involution of order n between the points of two algebraic surfaces, the author classifies such involutions according as the number of their coincidences is infinite or finite. The general characteristics of involutions of each of these two types are developed and some interesting special cases of each type discussed. Seventy-three papers are cited.

T. R. HOLLCROFT