

ON THE APPLICATION OF MARKOFF'S THEOREM  
TO PROBLEMS OF APPROXIMATION  
IN THE COMPLEX DOMAIN\*

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In a recent note,† the writer has discussed an extension to the complex domain of a method previously used in connection with problems of the approximate representation of real functions, in which the proof of convergence is based on Bernstein's theorem on the derivative of a polynomial or trigonometric sum. The discussion for the case of a complex variable involved certain restrictions on the boundary of the region with which the problem was concerned. The object of the present paper is to show how these restrictions may be somewhat lightened, at the expense, to be sure, of a compensating increase in the stringency of the hypotheses on the function to be approximated, if Markoff's theorem on the derivative of a polynomial is used in place of that of Bernstein.

Markoff's theorem can be stated for the purpose in hand as follows.

*If  $P_n(z)$  is a polynomial of the  $n$ th degree such that  $|P_n(z)| \leq L$  at all points of a line segment of length  $2h$  in the  $z$ -plane, then  $|P_n'(z)| \leq n^2L/h$  at all points of the same segment.*

The theorem is commonly stated for an interval of the axis of reals, and more particularly for the interval  $(-1, 1)$ , thus:‡ If  $|P_n(x)| \leq L$  for  $-1 \leq x \leq 1$ , then  $|P_n'(x)| \leq n^2L$  throughout the interval. If the hypothesis of the more general statement is given for the segment connecting the points  $z_1$  and  $z_2$ , the substitution  $z = \alpha + \beta z'$ , with  $\alpha = \frac{1}{2}(z_1 + z_2)$ ,  $\beta = \frac{1}{2}(z_2 - z_1)$ ,  $|\beta| = h$ , reduces one to the other. In the formulation for the interval  $(-1, 1)$ , to be sure, the coefficients are ordinarily thought of as

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† *On certain problems of approximation in the complex domain*, this Bulletin vol. 36 (1930), pp. 851-857.

‡ See, for example, Marcel Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354-368; pp. 359-360.

real, but this restriction is not necessary.\* For multiplication of the polynomial by a constant of modulus unity does not change either its absolute value or the absolute value of its derivative; and if the maximum  $M$  of the absolute value of the derivative is attained for  $x = x_0$  there is no loss of generality in assuming that a suitable constant factor has been introduced to give  $P'_n(x_0)$  itself the real value  $M$ . Then, if  $u_n(x)$  and  $iv_n(x)$  are the real and pure imaginary parts of  $P_n(x)$ ,  $u'_n(x_0) = M$  and  $v'_n(x_0) = 0$ , and since  $|u_n(x)| \leq |P_n(x)| \leq L$  throughout the interval, application of the theorem to the real polynomial  $u_n(x)$  gives  $M \leq n^2L$ .

Let  $R$  be any (finite) region of the plane† for which there is a positive number  $h$  such that from every point of the boundary a line segment of length  $2h$  can be drawn belonging wholly to the region. Let  $P_n(z)$  be a polynomial of the  $n$ th degree, and let  $L$  be an upper bound for its absolute value in  $R$ . (By the maximum-minimum theorem, together with the continuity of  $P_n(z)$ , the requirement in the last clause is fulfilled if  $|P_n(z)| \leq L$  throughout the interior of  $R$ , or, on the other hand, if  $|P_n(z)| \leq L$  everywhere on the boundary.) Under the conditions stated, every point of the boundary is a point of a line segment of length at least  $2h$ , on which  $|P_n(z)|$  does not exceed  $L$ . By Markoff's theorem, consequently,  $|P'_n(z)| \leq n^2L/h$  at every point of each of these segments, and in particular at every point of the boundary of  $R$ . From this it follows further that  $|P'_n(z)| \leq n^2L/h$  throughout the interior of  $R$ . The result may be summarized for reference as follows.

*If  $R$  is a region of the character specified above, and if  $P_n(z)$  is a polynomial of the  $n$ th degree such that  $|P_n(z)| \leq L$  throughout  $R$ , then  $|P'_n(z)| \leq kn^2L$  throughout  $R$ ,  $k$  being a constant which depends only on  $R$ .*

By way of commentary on the scope of the theorem, it may be noted that *the hypothesis with regard to  $R$  is satisfied if  $R$  is*

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\* The admissibility of complex coefficients is pointed out by Riesz, loc. cit. For a transition similar to that made here see also S. Bernstein, *Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle*, Paris, 1926, p. 45.

† In connection with the general problem of the extension of Markoff's theorem, as well as that of Bernstein, to the complex domain, see also G. Szegö, *Über einen Satz von A. Markoff*, *Mathematische Zeitschrift*, vol. 23 (1925), pp. 45–61; for the case of Bernstein's theorem on a circle in the complex plane, M. Riesz, loc. cit., p. 357, and S. Bernstein, op. cit., pp. 44–45.

any convex region bounded by a Jordan curve, or is made up of a finite number of such regions. For, in the case of a single convex region, if the boundary is divided into four segments in any way, the line joining any point of one segment to any point of the opposite segment has a positive minimum length, and belongs wholly to the region. The truth of the statement with regard to a composite region is an immediate corollary.

Now let  $R$ , whether meeting the terms of the last paragraph or not, be a region satisfying the original hypothesis as to the accessibility of all points of the boundary by line segments whose lengths have a positive lower bound, and let the boundary of  $R$  be a simple closed Jordan curve:

$$x = \phi(t), \quad y = \psi(t),$$

where  $\phi(t)$  and  $\psi(t)$  are continuous functions with period  $p$ , and

$$|\phi(t_2) - \phi(t_1)| + |\psi(t_2) - \psi(t_1)| \neq 0$$

for  $0 < |t_2 - t_1| < p$ . It would make no essential difference if  $R$  were a multiply connected region bounded by a finite number of such curves, or made up of a finite number of separate regions of similar character, but for simplicity the discussion will be formulated for the case of a single simply connected region.

Let  $f(z)$  be a function analytic throughout the interior of  $R$ , and continuous on the boundary. Let  $\rho(t)$  be a bounded and measurable function of period  $p$ , with a positive lower bound. Let  $P_n(z)$  be determined among all polynomials of the  $n$ th degree as the one for which the integral

$$\int_0^p \rho(t) |f(z) - P_n(z)|^m dt, \quad z = \phi(t) + i\psi(t),$$

has the smallest possible value,  $m$  being a given positive constant. This minimum problem has a solution, and for  $m > 1$  the solution is unique; the essentials of the proof, as applied to various problems of similar character, are well known.

The problem of approximation to which the above corollaries of Markoff's theorem are to be applied is that of the convergence of  $P_n(z)$  toward  $f(z)$  as  $n$  becomes infinite.

For this purpose one more assumption will be made with regard to the boundary of  $R$ , namely that the functions  $\phi(t)$ ,  $\psi(t)$  satisfy a Lipschitz condition:

$$|\phi(t_2) - \phi(t_1)| \leq \lambda |t_2 - t_1|, \quad |\psi(t_2) - \psi(t_1)| \leq \lambda |t_2 - t_1|.$$

The construction of a convergence proof then involves merely the adaptation of processes of reasoning which have been used repeatedly elsewhere.\*

Let  $p_n(z)$  be an arbitrary polynomial of the  $n$ th degree; let  $\epsilon_n$  be the maximum of  $|f(z) - p_n(z)|$  on the boundary of  $R$ ; let  $r_n(z) = f(z) - p_n(z)$ ; and let  $\pi_n(z) = P_n(z) - p_n(z)$ . Then

$$r_n(z) - \pi_n(z) = f(z) - P_n(z),$$

$$\gamma_n = \int_0^p \rho(t) |f(z) - P_n(z)|^m dt = \int_0^p \rho(t) |r_n(z) - \pi_n(z)|^m dt.$$

Let  $\mu_n$  be the maximum of  $|\pi_n(z)|$  on the boundary,  $z_1$  a point of the boundary at which this maximum is attained, and  $t_1$  the corresponding value of  $t$  in the interval  $(0, p)$ .

Markoff's theorem, as extended above, yields the fact that  $|\pi_n'(z)| \leq kn^2\mu_n$  throughout  $R$ . So

$$|\pi_n(z) - \pi_n(z_1)| = \left| \int \pi_n'(z) dz \right| \leq \int |\pi_n'(z)| ds \leq kn^2\mu_n\sigma,$$

if the integral is extended along any rectifiable path which goes from  $z_1$  to  $z$  without passing outside of  $R$ , and if  $\sigma$  is the length of this path. The boundary of  $R$  is rectifiable, by virtue of the Lipschitz condition on  $\phi$  and  $\psi$ . Hence it is readily deduced that  $|\pi_n(z)|$  remains greater than  $\frac{1}{2}\mu_n$  over an arc of the boundary whose length is of the order of  $1/n^2$ . By more specific application of the Lipschitz condition,  $z$  will remain on such an arc as  $t$  varies from  $t_1$  over an interval having a length of the order of  $1/n^2$ . Reasoning of the type set forth in the various passages referred to then shows that

$$|r_n(z) - \pi_n(z)| = |f(z) - P_n(z)| \leq cn^{2l}m\epsilon_n,$$

where  $c$  is independent of  $n$ .

For uniform convergence of  $P_n(z)$  toward  $f(z)$  in the interior and on the boundary of  $R$  it is sufficient that it be possible to choose polynomials  $p_n(z)$  for each value of  $n$  so that  $\lim_{n \rightarrow \infty} n^{2l}m\epsilon_n = 0$ .

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\* See, for example, D. Jackson, *On the convergence of certain trigonometric and polynomial approximations*, Transactions of this Society, vol. 22 (1921), pp. 158-166; *The Theory of Approximation*, New York, 1930, pp. 82-86, 96-98; this Bulletin, loc. cit.

The restriction on  $f(z)$  in this conclusion is more severe than in the earlier paper in this Bulletin, to the extent that the factor  $n^{1/m}$  in that paper has been replaced by  $n^{2/m}$ ; but the regions  $R$  admitted here are considerably more general. For example, the earlier result is not applicable to a rectangular region, which is admissible as a very special case under the present hypotheses. With respect to the question of uniform convergence throughout the given closed region, the present result is intermediate between the earlier one given by the author and those obtained by Walsh\* for still more general regions, under the hypothesis that the function represented is analytic on the boundary.

The scope of the present method can be somewhat broadened, as far as the character of the region is concerned, by generalizing the original statement of Markoff's theorem in other directions.

Attention will be given next to the case of trigonometric sums in a real variable.

Let  $C_n(\theta)$  be a cosine sum of the  $n$ th order, an expression of the form

$$a_0 + a_1 \cos \theta + \cdots + a_n \cos n\theta;$$

the coefficients may be real or complex. Let  $L$  be an upper bound for  $|C_n(\theta)|$  in the interval  $\gamma \leq \theta \leq \delta$ ,  $0 < \gamma < \delta < \pi$ . If  $x = \cos \theta$ ,  $C_n(\theta)$  is a polynomial of the  $n$ th degree in  $x$ ,  $P_n(x)$ , and  $|P_n(x)| \leq L$  for  $\cos \delta \leq x \leq \cos \gamma$ . Hence  $|P_n'(x)| \leq c_1 n^2 L$  throughout the same interval, if  $c_1$  stands for the quantity  $2/(\cos \gamma - \cos \delta)$ , the reciprocal of half the length of the interval over which  $x$  ranges, and

$$|C_n'(\theta)| = |P_n'(x)| \left| \frac{dx}{d\theta} \right| \leq c_1 n^2 L.$$

(Either of the extreme values  $\gamma = 0$ ,  $\delta = \pi$  could be admitted here equally well, but they are to be ruled out in the following paragraphs. If the hypothesis is satisfied for  $0 \leq \theta \leq \pi$ , it is satisfied for all values of  $\theta$ , and Bernstein's theorem is applicable.)

Let  $S_n(\theta)$  be a sine sum of the  $n$ th order, and let  $|S_n(\theta)| \leq L$  for  $\gamma \leq \theta \leq \delta$ , with  $0 < \gamma < \delta < \pi$  as before. It is possible to write  $S_n(\theta)$  as the product of  $\sin \theta$  and a cosine sum of order  $n-1$ , which may be denoted by  $C_{n-1}(\theta)$ , and the latter is a poly-

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\* J. L. Walsh, *On the overconvergence of sequences of polynomials of best approximation*, Transactions of this Society, vol. 32 (1930), pp. 794-816; *Note on the overconvergence of sequences of polynomials of best approximation*, the same Transactions, vol. 33 (1931), pp. 370-388.

mial of degree  $n - 1$  in  $x = \cos \theta$ . Let this polynomial be called  $P_{n-1}(x)$ . Then, throughout the range specified,

$$| P_{n-1}(x) | = | C_{n-1}(\theta) | = | S_n(\theta) / \sin \theta | \leq c_2 L,$$

if  $c_2$  is the larger of the quantities  $\csc \gamma$ ,  $\csc \delta$ . It follows that

$$\begin{aligned} | P'_{n-1}(x) | &\leq c_1 c_2 (n - 1)^2 L \leq c_1 c_2 (n^2 - 1) L, \\ | C'_{n-1}(\theta) | &= | P'_{n-1}(x) | | dx/d\theta | \leq c_1 c_2 (n^2 - 1) L, \\ | S'_n(\theta) | &= | \sin \theta C'_{n-1}(\theta) + \cos \theta C_{n-1}(\theta) | \\ &\leq c_1 c_2 (n^2 - 1) L + c_2 L \leq c_1 c_2 n^2 L, \end{aligned}$$

the last inequality being justified by the fact that  $c_1 > 1$ .

Now let  $T_n(\theta)$  be an arbitrary trigonometric sum of the  $n$ th order, and let  $| T_n(\theta) | \leq L$ , not only for  $\gamma \leq \theta \leq \delta$ , but also for  $-\delta \leq \theta \leq -\gamma$ . In the resolution

$$T_n(\theta) = \frac{1}{2} [ T_n(\theta) + T_n(-\theta) ] + \frac{1}{2} [ T_n(\theta) - T_n(-\theta) ],$$

the expressions in brackets are a cosine sum and a sine sum respectively, and each has  $2L$  as an upper bound for its absolute value in the interval  $\gamma \leq \theta \leq \delta$ . From the results of the two preceding paragraphs it is deduced that  $| T'_n(\theta) | \leq 2c_1 c_2 n^2 L$ . Since a cosine sum is an even function, and a sine sum is an odd function, this relation holds for the interval  $-\delta \leq \theta \leq -\gamma$  also.

As the substitution  $\theta' = \theta - \theta_0$  carries over a trigonometric sum of the  $n$ th order into an expression of the same form, the symmetry of the intervals  $(-\delta, -\gamma)$ ,  $(\gamma, \delta)$  with respect to the point  $\theta = 0$  has no significance; the essence of the conclusion is that if  $| T_n(\theta) | \leq L$  throughout two separate intervals of equal length,  $| T'_n(\theta) | \leq c_3 n^2 L$  throughout the same intervals, where  $c_3$  is a constant depending only on the length of the intervals and the distance between them.

To pass from this to an assertion of more general utility, let  $T_n(\theta)$  be an arbitrary trigonometric sum of the  $n$ th order, with real or complex coefficients, and let  $| T_n(\theta) | \leq L$  for  $\alpha \leq \theta \leq \beta$ , where  $\alpha$  and  $\beta$  have any values; it may be assumed without sacrificing anything in the result that  $\beta - \alpha < 2\pi$ , since otherwise Bernstein's theorem would be applicable. Let the interval  $(\alpha, \beta)$  be divided into four equal parts. The conclusion of the last paragraph may be applied to the first and third of these parts, and again to the second and fourth parts, to show that  $| T'_n(\theta) |$

$\leq c_3 n^2 L$  throughout the whole interval  $(\alpha, \beta)$ , where  $c_3$  has the appropriate value for the pairs of intervals indicated. In summary:

*If  $T_n(\theta)$  is a trigonometric sum of the  $n$ th order such that  $|T_n(\theta)| \leq L$  throughout a specified interval of length less than  $2\pi$ , then  $|T'_n(\theta)| \leq k'n^2L$  throughout the same interval,  $k'$  being a constant which depends only on the length of the interval.\**

To return to the complex plane, let  $P_n(z)$  be a polynomial of the  $n$ th degree, and  $L$  an upper bound for  $|P_n(z)|$  on an arc of the circle  $|z - z_0| = R$ , given by the specifications  $z - z_0 = Re^{i\theta}$ ,  $\alpha \leq \theta \leq \beta$ . On the circumference of the circle,  $P_n(z)$  reduces to a trigonometric sum of the  $n$ th order in  $\theta$ ,  $T_n(\theta)$ . For  $\alpha \leq \theta \leq \beta$ , furthermore,  $|T_n(\theta)| \leq L$ . Consequently

$$|P'_n(z)| = |T'_n(\theta)|/R \leq k'n^2L/R.$$

Hypothesis and conclusion may be put together as follows:

*If  $P_n(z)$  is a polynomial of the  $n$ th degree such that  $|P_n(z)| \leq L$  at all points of a circular arc in the  $z$ -plane, then  $|P'_n(z)| \leq Kn^2L$  at all points of the same arc, where  $K$  is a constant depending only on the radius and length of the arc; for arcs of equal angular measure  $K$  may be taken inversely proportional to the radius.*

It is possible now to formulate immediately a lemma of similar character, and to proceed thence to a proof of convergence of sequences of approximating polynomials, for certain two-dimensional regions to which the earlier reasoning based on Markoff's theorem for a rectilinear segment would not have been applicable; for example, a region bounded in part by two circular arcs tangent to each other internally. Space will not be taken here to give the statements at length.

The analog of Markoff's theorem can be extended directly to arcs somewhat more general than circular arcs, namely to any arc given by equations of the form  $x = \phi(t)$ ,  $y = \psi(t)$ , where  $\phi$  and  $\psi$  are polynomials or trigonometric sums such that

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\* For a corresponding corollary of Bernstein's theorem, see D. Jackson, *A generalized problem in weighted approximation*, Transactions of this Society, vol. 26 (1924), pp. 133-154; pp. 139-145. While there is no mention of complex coefficients in the passage cited, it is obvious that the admission of such coefficients raises a question only as to the value of the constant in the right-hand member of the inequality, and is trivial as long as the constant is left unspecified.

$|\phi'(t)| + |\psi'(t)| > 0$  over the range of variation of  $t$  (inclusive of its end points). For if  $P_n(z)$  is a polynomial of the  $n$ th degree it is expressible on the curve as a polynomial or trigonometric sum in  $t$ , whose degree or order is not greater than a certain constant multiple of  $n$ ; and as  $|dz/dt|$  is continuous and positive on the closed interval of values of  $t$ , the absolute value of the derivative of  $P_n(z)$  with respect to  $z$  will not exceed a constant multiple of the absolute value of its derivative with respect to  $t$ . A simple example is an arc of the ellipse  $x = a \cos t$ ,  $y = b \sin t$ .

If  $\phi(t)$  and  $\psi(t)$  are trigonometric sums, and if one considers the closed curve described as  $t$  ranges over a whole period, it is possible to derive similarly an analog of Bernstein's theorem, with a factor  $n$  in place of the  $n^2$  of Markoff's theorem. The result thus obtained is much less general than that given in the author's earlier paper\* with respect to the form of the functions  $\phi$  and  $\psi$ , but possesses the advantage that the curve may have double points without introducing any additional complication.

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\* This Bulletin, loc. cit., p. 856.