A NOTE ON PRIMITIVE IDEMPOTENT ELEMENTS OF A TOTAL MATRIC ALGEBRA*

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We consider a total matric algebra M over a field F, whose general element is

$$u = \sum \alpha_{ij} e_{ij}, \qquad (i, j = 1, \cdots, n),$$

where $e_{ij}e_{lk} = e_{ik}$ if $j = l$, and $e_{ij}e_{lk} = 0$ for $j \neq l$.

THEOREM 1. *A necessary and sufficient condition that $u = \Sigma\alpha_{ij}e_{ij}$ be idempotent in M is*

$$(1) \qquad \sum_s \alpha_{ps}\alpha_{sq} = \alpha_{pq}, \qquad (p, q = 1, \cdots, n).$$

This is seen immediately on writing

$$u^2 = \sum_{p,q,s} \alpha_{ps}\alpha_{sq}e_{pq}$$

and comparing with

$$u = \sum_{p,q} \alpha_{pq}e_{pq}.$$

THEOREM 2. *A necessary and sufficient condition for an idempotent element u to be primitive in M is*

$$(2) \qquad \alpha_{pi}\alpha_{jq} = \alpha_{pq}\alpha_{ji}, \qquad (p, q, i, j = 1, \cdots, n).$$

For, let

$$u = \sum_{r,t} \alpha_{rt}e_{rt}$$

be a primitive indempotent element of M. Let $\alpha_{ji} \neq 0$. Then the element $ue_{ij}u/\alpha_{ji}$ is idempotent in uMu, since

$$\left(\frac{ue_{ij}u}{\alpha_{ji}}\right)^2 = \frac{u \cdot e_{ij}ue_{ij} \cdot u}{\alpha_{ji}^2} = \frac{u\alpha_{ji}e_{ij}u}{\alpha_{ji}^2} = \frac{ue_{ij}u}{\alpha_{ji}}.$$

Hence† we have $ue_{ij}u/\alpha_{ji} = u$, and $ue_{ij}u = \alpha_{ji}u$. Equating coef-

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† Dickson, *Algebras and their Arithmetics*, p. 55.

ficients of e_{pq}, we have $\alpha_{pi}\alpha_{jq}=\alpha_{pq}\alpha_{ji}$. Conversely, suppose that

$$u=\sum\alpha_{rt}e_{rt}$$

is idempotent in M and let

$$\alpha_{pi}\alpha_{jq}=\alpha_{pq}\alpha_{ji}, \qquad (p, q, i, j=1, \cdots, n).$$

It follows that u is a primitive idempotent. In proof, let

$$m=\sum\beta_{sp}e_{sp}$$

be any element of M. Then

$$umu = \sum_{r,s,p,t} \alpha_{rs}\beta_{sp}\alpha_{pt}e_{rt} = \sum_{r,s,p,t} \alpha_{ps}\beta_{sp}\cdot\alpha_{rt}e_{rt}.$$

The coefficient of $\alpha_{rt}e_{rt}$ is

$$\sum_{s,p}\alpha_{ps}\beta_{sp},$$

which is independent of r and t. It follows that

$$umu = \Big(\sum_{s,p}\alpha_{ps}\beta_{sp}\Big)\Big(\sum_{r,t}\alpha_{rt}e_{rt}\Big) = \sum_{s,p}\alpha_{ps}\beta_{sp}\cdot u.$$

This shows that the algebra uMu is of order 1, based on its modulus.* Hence u is a primitive idempotent element of M.

Applying (2) to (1), we obtain the following corollary.

COROLLARY. *Relations* (2) *and* (3) *below constitute necessary and sufficient conditions for u to be a primitive idempotent element of M*:

$$\left.\begin{aligned} &(2) \qquad \alpha_{pi}\alpha_{jq} = \alpha_{pq}\alpha_{ji} \\ &(3) \qquad \sum\alpha_{pp} = 1 \end{aligned}\right\}, \qquad (p, q, i, j = 1, \cdots, n).$$

The above Corollary leads to a simple device for obtaining the coordinates of a primitive idempotent element of a total matric algebra of order n. Construct a square array of n rows and n columns, as follows.

First write numbers, $\alpha_{ii}(i=1, \cdots, n)$ along the principal diagonal so that $\Sigma\alpha_{ii}=1$. Then write in numbers α_{i1} arbitrarily in the first column. Next write in numbers α_{ij} chosen to satisfy the relation $\alpha_{i1}\alpha_{jj}=\alpha_{ij}\alpha_{j1}$, $(i=1, \cdots, n;\ j=2, \cdots, n)$. It follows readily that $\alpha_{ij}\alpha_{km}=\alpha_{im}\alpha_{kj}$, $(i, j, k, m=1, \cdots, n)$.

We illustrate with the array

* Nowlan, *On the direct product of a division and a total matric algebra*, this Bulletin, vol. 36 (1930), p. 267, Theorem 5.

$$\begin{pmatrix} \frac{1}{3} & 1 & 0 \\ \frac{2}{9} & \frac{2}{3} & 0 \\ -2 & -6 & 0 \end{pmatrix}$$

for the case $n=3$. This gives the primitive idempotent element

$$u = \tfrac{1}{3}e_{11} + e_{12} + \tfrac{2}{9}e_{21} + \tfrac{2}{3}e_{22} - 2e_{31} - 6e_{32}.$$

We define supplementary primitive idempotent elements as follows.

DEFINITION. A set of primitive idempotent elements is said to be *supplementary* in case their sum equals the modulus and if, further, the product of each pair in either order is zero.*

We now determine necessary and sufficient conditions that a set of primitive idempotent elements shall be supplementary.

Let u_i and $u_j (i \neq j)$ be two of a set of supplementary primitive idempotent elements. A necessary and sufficient condition for the relation $u_i u_j = u_j u_i = 0$, is obviously

$$\sum_s \alpha_{rs}^{(i)} \alpha_{st}^{(j)} = 0,$$

where $\alpha_{rs}^{(i)}$ and $\alpha_{st}^{(j)}$ are the general coordinates of u_i and u_j respectively. Combining this result with the condition that the sum of the components of a set of supplementary primitive elements shall equal the modulus, we have the following result.

THEOREM 3. *Equations* (4), (5) *and* (6), *which follow, constitute necessary and sufficient conditions that a set of primitive idempotent elements shall be supplementary*:

$$\sum_s \alpha_{rs}^{(i)} \alpha_{st}^{(j)} = 0, \qquad (i \neq j \text{ and } r, s, t, i, j = 1, \cdots, n); \tag{4}$$

$$\sum_{k=1}^{n} \alpha_{rl}^{(k)} = 0, \qquad (r \neq l \text{ and } r, l = 1, \cdots, n); \tag{5}$$

$$\sum_{k=1}^{n} \alpha_{rr}^{(k)} = 1, \qquad (r = 1, \cdots, n). \tag{6}$$

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* A supplementary set of primitive idempotent elements of a total matric algebra of order n^2 contains exactly n elements. See F.S. Nowlan, this Bulletin, vol. 36 (1930), p. 268.