LEFSCHETZ ON TOPOLOGY

Topology. By Solomon Lefschetz. New York (American Mathematical Society Colloquium Publications, Volume 12), published by this Society, 1930. 409 pp.

The rapid growth of interest in the science of analysis situs has been one of the striking facts of recent mathematical history. Of decided importance in the development of that branch of the science which deals with n-dimensional manifolds was the publication in 1922 of Veblen's Colloquium Lectures. In this volume the connectivity theory, one of the finest creations of Poincaré, was put onto the rigorous foundations which had been prepared for it in the work of Veblen and Alexander, and the many problems which the more exacting viewpoint entailed were clearly brought into evidence. The "speedy obsolescence" which Veblen wished for his book has by no means taken place; but the success of his work is evident from the many important advances to which it has given stimulus. One of the outstanding leaders of this recent progress is the author of the present volume. His researches have extended over a period of several years, and have now been brought to a brilliant culmination in this latest addition to the Colloquium Publication series.

Almost the whole of Lefschetz's work in analysis situs has centered around the following fundamental problem: a given space is subjected to a continuous transformation into itself; how can a topologically significant index be attached to the totality of fixed points and how can its value be computed in terms of the invariants of the situation? In an early series of papers on surface transformations, Brouwer included the first explicit solution of this problem for a very special case. If a sphere is subjected to a single-valued sense-preserving transformation into itself, then, as Brouwer pointed out, the algebraic sum of the multiplicities of the fixed points is always two, and from this the existence of at least one fixed point is immediately inferred. Other authors treated somewhat more general cases and a number of interesting results were obtained, notably by Birkhoff and Alexander. It was Lefschetz, however, who in 1923 proposed a method of attack which penetrated into the heart of the problem in its most general form, and brought it immediately within working range of the tools of n-dimensional analysis situs as they had been developed in Veblen's lectures. If M' is a copy of a given space M, let us form, said Lefschetz, the product space $M\times M'$. Then a transformation T of M may be thought of as a pairing off of the points of M and M', and the totality of point-pairs will constitute a subspace V of $M\times M'$. The identity in particular will lead to subspace V_0, and the fixed points of T will then correspond to the intersections of the two subspaces V and V_0. Thus the concept of transformation, with its connotation of change, is essentially a static geometric situation, and as such can most advantageously be studied.

This, then, is the guiding principle for Lefschetz's attack on the problem, and it is interesting to observe the fruitfulness of an idea of such simplicity and intuitive content. There is hardly an important situation which does not yield to its power. Not only may the manifolds be of arbitrary dimension and structure, with or without boundary, but the transformations themselves may be of almost the greatest possible generality consistent with the word *continuous.* The author goes even further, and in a highly interesting chapter on infinite complexes, shows how his methods can be applied to arbitrary compact metric spaces. That a theorem such as the following is a very special corollary indicates the range of the results. Suppose a compact metric space to be simply connected in every order, as well as locally connected; then every continuous single-valued transformation of the space into itself has at least one fixed point.

It must not be supposed that the present volume deals exclusively with a single problem. On the contrary, it is rather a complete and skillful exposition of the analysis situs in which the problem is immersed, and this means nearly the whole of the theory of complexes as it now exists. The foundations of the combinatorial part are firmly laid at the outset. There follow the fundamental invariance proofs. In the chapter on manifolds, the introduction and systematic use of *combinatorial cells,* which like many other fundamental ideas in this connection are due to Alexander, is particularly noteworthy. For in terms of these cells it becomes possible for the first time to formulate a combinatorial definition of manifold which is topologically invariant. The author's proof of this, following somewhat along the lines of the first published one, that of van Kampen, is remarkably compact, thanks to the use of the author's convenient relative concepts, to which we shall again refer. In this same chapter, the author rounds out his recent interesting results with regard to duality relations. There are two well known types of such relations: those discovered by Poincaré which exist between the various connectivity indices of a manifold, and those due to Alexander in which the invariants of a surrounding residual space also enter. The author's discovery that these two types of relations are special cases of a third more general type, is revealed in a set of formulas of striking symmetry and generality.

The remainder of the book contains the material which is most closely connected with the author's own researches,—the theory of intersections and Kronecker indices, product complexes, fixed point and coincidence formulas. The majority of the results here are the author's own, and many of them are already in the literature. In its complete systematic development, however, this part of the theory has been considerably strengthened; the results are now more general and many of the proofs much simpler. In order to carry over the main body of his results to a much more general class of spaces than he had previously considered, the author has included a new general theory of infinite complexes. The fact that an arbitrary open n-dimensional region is a very special type of infinite complex, is an indication of the types of spaces which this theory brings into play. The author obtains a complete classification of the various sorts of homology bases which can occur in the infinite case, and he is able to extend his earlier duality theory by the ingenious introduction of "ideal" elements. By means of the far reaching approximation methods of Alexandroff and Vietoris we are led to a homology theory for arbitrary compact metric spaces, and it is at this point that the duality and transformation theories receive their final extensions.

The book closes with a summary of the principal applications of analysis situs to function theory and geometry. It is no mere accident that the foregoing theory can be so admirably applied to the solution of certain difficult problems in algebraic geometry,—those, for example, which concern the enumeration of the intersections of algebraic varieties and coincidences of algebraic correspondences. To some extent it is for these very problems that the theory has been created. They have especially influenced the author's mathematical work. He is known as an important contributor to algebraic geometry, and he writes here with particular authority. There are included also the applications to vector distributions, complex spaces, and the Kronecker characteristic theory. In all these questions, it is of course important to know that the analytically defined manifolds with which one has to deal are complexes in the technical sense,—that is, capable of being subdivided into cells. It has long been regarded as certain that such is the case, although, except for the relatively simple algebraic case, no complete proof exists. In sketching a possible proof for the general case, the author shows that he himself has no doubts about the matter. It may be remarked, however, that there is even less doubt that a thorough exposition of the details involved is a formidable task in function theory, and one that will be thoroughly worth undertaking.

From a technical point of view, the book contains much that is new and valuable. The simple device of neglecting, in a set of boundary relations, all elements which lie on a given configuration L, thus giving rise to relations "mod L," more than justifies itself by its usefulness, and will undoubtedly become a standard part of the combinatorial technique. By means of this device we are led immediately to the concept of relative invariants, and it is in terms of these that many of the author's most striking formulas are expressed. Moreover, it becomes unnecessary ever to treat separately the manifolds with boundaries and the theorems concerning closed manifolds now become corollaries of the more general ones. Worth noting also is the neat explicit statement of the fundamental deformation theorem and its various extensions. This theorem was implied in Alexander's original proof of the invariance of the homology characters and again in the same connection by Veblen in his Lectures. Its utility is almost universal, for in any given topological situation it permits us to limit our attention to the configurations composed strictly of the cells of the given spaces. How extensively Alexander's elegant new technique will be applicable, remains still to be seen. The present work, however, makes it quite clear that the older deformation method, on account of its simplicity and intuitiveness, is not likely soon to outlive its usefulness.

In the matter of presentation, we feel that the author has done an excellent piece of work. In a subject where the formal element plays so small a part, the writing is often difficult. While Lefschetz's exposition is necessarily compact and requires close attention on the part of the reader, it will be found to be accurate* and essentially complete. It adds no little to the work that the various

* The book is remarkably free from misprints. The following minor corrections have been suggested by the author:
page 3, line 11, replace "of R" by "for $T\cdot R$"
" 80, " 3, " "$T\cdot \quad C$" by "$T\cdot C$"
" 91, " 18, 19, " "regular" by "normal"
" 122, between 3d and 4th lines from bottom, insert: (see page 648)

clerical details which can be of considerable aid to the reader,—the careful numbering of paragraphs and formulas, adequate cross-references, and the explicit statement of theorems,—have been scrupulously attended to. There are numerous diagrams and an excellent bibliography.

Analysis situs is a comparatively young science; yet its importance can not be doubted. It deals with the most primitive questions of geometry and is fascinating to those who enjoy moulding in precise mathematical rigor the visions of a sharpened physical intuition. Its most important problems deal with the very structure of space, and many of them are still to be solved. It is pleasant to reflect that much of what has been accomplished has been the work of American mathematicians, and to that work the present volume is a distinguished contribution.

P. A. SMITH

FUBINI AND ČECH

Introduction à la Géométrie Projective Différentielle des Surfaces. By Fubini et Čech. Paris, Gauthier-Villars, 1931. vi+291 pp.

This volume is in some sense a sequel to the treatise entitled *Geometria Proiettiva Differenziale* published in two volumes in 1926 and 1927 by the same authors. It should receive a generous welcome from the geometrical public for several reasons. First of all, it is in French, a language admittedly more widely read than Italian. The authors, profiting no doubt by their previous experience, have produced a quite readable book. Some detailed developments of their treatise have been omitted; the treatment here is more elementary, and the style of exposition is clearer than before. The discussion is confined to surfaces in ordinary space. Altogether, this book should serve well its purpose of being an introduction to projective differential geometry.

It must not be understood that the present volume is merely an abstract from the treatise. In fact, certain subjects are included which do not appear in the larger work at all. An analysis of the contents of the volume under review will amplify these remarks.

There are in all fourteen chapters, of which the first three may be regarded as introductory. The first is properly an introduction, containing some preliminary analytical results concerning collineations and correlations, matrices, and algebraic forms. The second treats of plane curves, and the third of curves in ordinary space.

Chapters IV–VIII contain an exposition of the projective differential theory of curved surfaces in ordinary space. The point of view is primarily that of the method of differential forms, but the differential equations which define a

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(Footnote contined from page 647.) "The proof as outlined holds for simplicial and convex cells, the only types for which it is used later in the book. The proof for the general case has been obtained recently by A. W. Tucker." Chapter III, No. 45, replace everywhere "M" by "$\overline{M}$."
page 206, end of No. 49, add, "For $n=1$, $Lc\,(\Gamma_0\cdot\Gamma_0')$ can also be defined provided that either Γ_0 or $\Gamma_0'\sim 0$.
page 403, end of reference to W. Mayer, add "219–258."