A REMARK CONCERNING THE NECESSARY CONDITION OF WEIERSTRASS*

BY E. J. MCSHANE†

Let us consider a class $\mathfrak{K}$ of rectifiable curves C lying in a point set A of n-dimensional space, and an integral $F(C) = \int_C f(x, x')ds$, where $x = (x^1, \cdots, x^n)$ and s connotes that we use the length of arc as parameter. Suppose that a certain curve $C: x = x(s)$ minimizes $F(C)$ in $\mathfrak{K}$, and denote by L the set of points of C which are interior to A and of indifference with respect to $\mathfrak{K}$ and A. Then for almost all points of L we have‡ $E(x(s), x'(s), \bar{x}') \geqq 0$ for all sets of numbers $\bar{x}'$. Given now a particular point $x(s_0)$ of L; when can we say that the inequality holds at $x(s_0)$?

It has already been shown§ that the inequality holds if $x'(s_0)$ exists, $\Sigma[x^{i\prime}(s_0)]^2 > 0$, and the $x^{i\prime}(s)$ are all approximately continuous at s_0. We will now show that the inequality also holds if $\Sigma(x^{i\prime}(s_0))^2 = 1$. (As is well known, this sum never exceeds 1, and is equal to 1 almost everywhere.)

Suppose then that $\Sigma[x^{i\prime}(s_0)]^2 = 1$ and that in contradiction to our statement there exists an $\bar{x}'$ such that $E(x(s_0), x'(s_0), \bar{x}') = -2k < 0$. Denote by $\alpha(s)$ the angle between $x'(s)$ and $x'(s_0)$. The function

$$\begin{aligned}\phi(s) &= \frac{d}{ds}\left[\sum x^i(s)x^{i\prime}(s_0)\right] = \sum x^{i\prime}(s)x^{i\prime}(s_0)\\ &= \left\{\sum [x^{i\prime}(s)]^2\right\}^{1/2}\left\{\sum [x^{i\prime}(s_0)]^2\right\}^{1/2}\cos\alpha(s)\end{aligned}$$

is defined for almost all values of s, and $|\phi(s)| \leqq |\cos\alpha(s)|$. By the continuity of E, we can find positive numbers ϵ, δ such that $E(x(s), x'(s), \bar{x}') < -k$ for all s such that $|s - s_0| \leqq \epsilon$, $\phi(s) \geqq 1 - \delta$; and if ϵ be small enough, $x(s)$ will be in L. But $\phi(s_0) = 1$ and $\phi(s)$

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‡ L. Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. 2, p. 87. E. J. McShane, *On the necessary condition of Weierstrass*, etc., Annals of Mathematics, vol. 32.

§ E. J. McShane, loc. cit.

is a derivative; therefore* there exists on $[s_0-\epsilon, s_0+\epsilon]$ a set of positive measure for which $\phi(s)>1-\delta$, "..., which contradicts the theorem quoted above."

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A CORRECTION AND AN ADDITION

BY G. E. RAYNOR

1. *A Correction.* In a former paper† by the author the minus sign on the right side of equation (4), page 888, makes the notations of equations (4) and (5) for the function G inconsistent. This difficulty may be removed by changing the sign of G throughout the paper wherever the first argument of G has r_1 in the denominator. This change makes the first footnote on page 888 superfluous and it should be deleted. The second argument of G in equations (9) and (20) should be 0 instead of θ.

2. *An Addition.* The mean value of the function Φ over the circle C_2 was considered, in the paper, for the case of the singular point P outside of C_2 and for the case of P inside of C_2. The question naturally arises as to what the situation is in case P lies on C_2. This third case is not, however, of much interest since the integral

$$\int_{C_2} \Phi ds,$$

which is now in general improper, will not in general exist. This may readily be verified for the function

$$\Phi = \left(\frac{r^2}{r_1^2} - \frac{r_1^2}{r^2}\right) \cos 2\theta$$

integrated over the circle C_2, whose equation is $\rho = r_1 \sin\theta$. It will be found that even the principal value of the above integral is infinite while of course the value of Φ at the center of C_2 is finite.

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* Hobson, *Theory of Functions of a Real Variable*, vol. 1, §403.

† *On the extension of the Gauss mean-value theorem to circles in the neighborhood of isolated singular points of harmonic functions*, this Bulletin, vol. 36 (1930), pp. 887–893.