NOTE ON FUNCTIONAL FORMS QUADRATIC IN A FUNCTION AND ITS FIRST p DERIVATIVES*

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It has been shown that a normal functional form quadratic in a function and its derivative is reducible to an invariantive form quadratic in a function and a constant, and that the continuity of the coefficients of the first form implies the continuity of those of the second.† The result is here generalized to a form quadratic in a function and p derivatives.

Let $y_0{}^\alpha$ denote a continuous function of the real variable α defined on the range $a \leqq \alpha \leqq b$. Let $y_1{}^\alpha, y_2{}^\alpha, \cdots, y_p{}^\alpha$ denote the corresponding derivatives defined and continuous on the same interval. The normal quadratic form is

$$Q = \sum_{i,j=0}^{p} A_{\alpha\beta}^{ij} y_i^{\alpha} y_j^{\beta} + A_{\alpha}^{ij} y_i^{\alpha} y_j^{\alpha}, \tag{A}$$

where $A_{\alpha}{}^{i}{}_{\beta}{}^{j}$ and $A_{\alpha}{}^{ij}$ are functions integrable in the Riemann sense, and where a Greek index repeated as a subscript and as a superscript is understood to denote integration with respect to that variable over the fundamental interval (a,b). The form Q is such that no generality is lost by assuming that $A_{\alpha}{}^{i}{}_{\beta}{}^{j} = A_{\beta}{}^{j}{}_{\alpha}{}^{i}$ and $A_{\alpha}{}^{ij} = A_{\alpha}{}^{ji}$. This assumption is made.

The convention of denoting integration by repeated Greek indices will be used in all that follows. In addition, when a Latin index (except p) occurs in a single term once as a superscript and once as a subscript, we shall understand summation with respect to that index over an integer range $(0, p-1)$. Thus

$$\begin{aligned} Q = {} & A_{\alpha\beta}^{ij} y_i^{\alpha} y_j^{\beta} + A_{\alpha\beta}^{pj} y_p^{\alpha} y_j^{\beta} + A_{\alpha\beta}^{ip} y_i^{\alpha} y_p^{\beta} + A_{\alpha\beta}^{pp} y_p^{\alpha} y_p^{\beta} \\ & + A_{\alpha}^{ij} y_i^{\alpha} y_j^{\alpha} + A_{\alpha}^{pj} y_p^{\alpha} y_j^{\alpha} + A_{\alpha}^{ip} y_i^{\alpha} y_p^{\alpha} + A_{\alpha}^{pp} (y_p^{\alpha})^2. \end{aligned} \tag{A'}$$

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* Presented to the Society, December 30, 1930.

† A. D. Michal and L. S. Kennison, *Quadratic functional forms on a composite range*, Proceedings of the National Academy of Sciences, vol. 16 (1930), pp. 617–619.

We now state and prove the following theorems:

THEOREM 1. *The form Q may be written as a form quadratic in the single function* $w^\alpha = y_p^\alpha$ *and the p constants* $y^i = y_i^a$, $(i=0, 1, \cdots, p-1)$.

The reduction is readily accomplished by use of a remainder form of Taylor's theorem:

$$(1) \qquad \overset{\alpha}{y_i} = \overset{a}{y_i} + (\alpha - a)\overset{a}{y}_{i+1} + \cdots + \frac{(\alpha - a)^{p-i-1}}{(p-i-1)!}\overset{a}{y}_{p-1} + \int_a^\alpha \frac{(\alpha-\sigma)^{p-i-1}}{(p-i-1)!} y_p{}^\sigma d\sigma.$$

However, it is convenient to condense this by defining the functions

$$\underset{i}{\overset{\alpha}{E}}{}_\sigma = \frac{(\alpha-\sigma)^{p-i-1}}{(p-i-1)!}, \qquad \text{if } \alpha > \sigma,$$

$$= \frac{1}{2}\,\frac{(\alpha-\sigma)^{p-i-1}}{(p-i-1)!}, \text{ if } \alpha = \sigma,$$

$$= 0, \qquad \text{if } \alpha < \sigma;$$

$$\underset{i}{\overset{\alpha}{E}}{}_k = \frac{(\alpha-a)^{k-i}}{(k-i)!}, \qquad \text{if } k \geqq i,$$

$$= 0, \qquad \text{if } k < i.$$

The equation (1) then becomes simply

$$(2) \qquad \overset{\alpha}{y_i} = \underset{i}{\overset{\alpha}{E}}{}_k \overset{k}{y} + \underset{i}{\overset{\alpha}{E}}{}_\sigma \overset{\sigma}{w}.$$

If the latter result now be placed in the expression (A′) for Q, we have

$$(\mathrm{B}) \qquad Q = B_{kl}y^k y^l + 2B_{k\sigma}y^k w^\sigma + B_{\sigma\tau}w^\sigma w_\tau + B_\sigma(w^\sigma)^2,$$

where

$$
(3)\quad
\begin{aligned}
B_{kl} &= A_{\alpha\beta}^{ij}\underset{i}{E}{}_k^{\alpha}\underset{j}{E}{}_l^{\beta} + A_{\alpha}^{ij}\underset{i}{E}{}_k^{\alpha}\underset{j}{E}{}_l^{\alpha},\\
B_{k\sigma} &= A_{\alpha\beta}^{ij}\underset{i}{E}{}_k^{\alpha}\underset{j}{E}{}_\sigma^{\beta} + A_{\alpha}^{ij}\underset{i}{E}{}_k^{\alpha}\underset{j}{E}{}_\sigma^{\alpha} + A_{\alpha\sigma}^{ip}\underset{i}{E}{}_k^{\alpha} + A_{\sigma}^{ip}\underset{i}{E}{}_k^{(\sigma)},\\
B_{\sigma\tau} &= A_{\alpha\beta}^{ij}\underset{i}{E}{}_\sigma^{\alpha}\underset{j}{E}{}_\tau^{\beta} + A_{\alpha}^{ij}\underset{i}{E}{}_\sigma^{\alpha}\underset{j}{E}{}_\tau^{\alpha} + 2A_{\alpha\sigma}^{ip}\underset{i}{E}{}_\tau^{\alpha} + A_{\tau}^{pj}\underset{j}{E}{}_\sigma^{(\tau)} + A_{\sigma}^{ip}\underset{i}{E}{}_\tau^{(\sigma)}\\
&\qquad + A_{\sigma\tau}^{pp},\\
B_{\sigma} &= A_{\sigma}^{pp},
\end{aligned}
$$

where the indices in parentheses indicating suspension of the integration until the coefficient B has been associated with the function w. Thus we demonstrate the theorem by giving an explicit expression for the new coefficients B in terms of the coefficients A.

THEOREM 2. *If the coefficients A are continuous, the coefficients B as given by* (3) *are also continuous.*

This may be seen at once from the expressions (3). The coefficients B are the sums of certain of the coefficients A and a finite number of definite integrals of the products of the coefficients A by the functions E. Since the functions E are all continuous, except ${}_{p-1}E_{\sigma}^{\alpha}$, any discontinuities of B must arise in terms containing ${}_{p-1}E_{\sigma}^{\alpha}$. Consider then these terms. Terms of the type $A_{\alpha}{}^{ip-1}{}_iE_k^{\alpha}{}_{p-1}E_{\sigma}^{\alpha}$ in which ${}_{p-1}E_{\sigma}^{\alpha}$ appears with one of its variables in integrated form must be continuous, for they are simply integrals in which the non-integrated variable of ${}_{p-1}E_{\sigma}^{\alpha}$ is one of the limits. For example,

$$A_{\alpha}^{ip-1}\underset{i}{E}{}_k^{\alpha}\underset{p-1}{E}{}_\sigma^{\alpha} = \int_{\sigma}^{b}(A_{\alpha}^{ip-1}\underset{i}{E}{}_k^{(\alpha)})d\alpha.$$

There is a possible exception in the term

$$A_{\alpha}{}^{p-1p-1}\underset{p-1}{E}{}_\sigma^{\alpha}\underset{p-1}{E}{}_\sigma^{\alpha},$$

but we see that

$$A_{\alpha}^{p-1\ p-1}\underset{p-1}{E}{}_\sigma^{\alpha}\underset{p-1}{E}{}_\tau^{\alpha} = \int_{\mu(\sigma,\tau)}^{b}(A_{\alpha}^{p-1\ p-1})d\alpha,$$

where μ is equal to the greater of σ and τ. Being a continuous function of a continuous function, the term is continuous. The only terms in which ${}_{p-1}E_\sigma^\alpha$ appears unintegrated are the terms $A_\sigma{}^{p-1p}{}_{p-1}E_\gamma{}^{(\sigma)}$ and $A^{p}{}_{\tau}{}^{p-1}{}_{p-1}E_\sigma{}^{(\gamma)}$. However, their sum is equal to $A_\mu{}^{p-1p}$ where μ is defined as above. Thus all the coefficients B are continuous,if the coefficients A are.

It has been shown elsewhere* that the form (B) of Q is invariantive; that is, a necessary and sufficient condition that Q vanish for all continuous functions w^α for which $w^a = w^b = 0$ and for all constants y^k is that the coefficients B be zero. Thus we may state

THEOREM 3. *A necessary and sufficient condition that a form quadratic in a function and its first p derivatives vanish for all functions $y_0{}^\alpha$ whose pth derivative is continuous and vanishes at the end points is that the coefficients A satisfy the equations obtained by equating to zero the right members of the equations* (3).

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* See A. D. Michal, American Journal of Mathematics, 1928; A. D. Michal and T. S. Peterson, forthcoming issue of the Annals of Mathematics; A. D. Michal, Proceedings of National Academy, 1930; A. D. Michal, forthcoming issue of American Journal of Mathematics; A. D. Michal and L. S. Kennison, loc. cit.