ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

114. Dr. W. Cauer: *Electric networks and bounded functions.*

The problem of designing communication networks with prescribed external characteristics corresponds essentially to the following: Given a complex variable λ, three positive quadratic forms of n variables with matrices L, R, D; to find the conditions that a given minor matrix of m-th order of functions is a chief minor of some A^{-1} where $A=\lambda L+R+\lambda^{-1}D$; then to find one or all sets of the original L, R, D which produce the given minor. A^{-1} is regular in the whole right half-plane, real for real λ, and the real part of A^{-1} belongs to a positive definite quadratic form whenever λ is a point of the right half-plane. The converse, that under these conditions one may construct positive definite L, R, D producing the given minor, seems to be true in general. This paper proves this theorem for: (1) $m=1$ (a) under restriction of degree of the driving point impedence corresponding to the R. M. Foster case (Bell System Technical Journal, 1924, p. 651), but $n=2$ or 3, (b) n infinite. (2) $m=2$, minor symmetric with respect to the secondary diagonal (symmetrical four-terminal network). Graphically given minors are considered. (Received December 20, 1930.)

115. Dr. W. Seidel (National Research Fellow): *On the approximation of continuous functions by linear combinations of continuous functions.*

The author presents a new proof of the following theorem of F. Riesz: *Let $\phi_1(x)$, $\phi_2(x)$, $\cdots$, $\phi_n(x)$, $\cdots$ be a sequence of arbitrary, linearly independent functions, defined and continuous in the interval $a\leqq x\leqq b$. A necessary and sufficient condition that linear combinations of these functions uniformly approximate every function $f(x)$, defined and continuous in $a\leqq x\leqq b$, is that there shall exist no function $\alpha(x)$, of bounded variation, satisfying the system of integral equations $\int_a^b\phi_i(x)d\alpha(x)=0$, $i=1, 2, \cdots$, except if $\alpha(x)$ is constant in the interval $a\leqq x\leqq b$ save perhaps for a denumerable set of values of x different from a and b.* Consider an $(n+1)$-dimensional euclidean space $(x_0, x_1, \cdots, x_n)$ and define the two curves $\Gamma_n^+ : x_0=f(x)$, $x_1=\phi_1(x)$, $\cdots$, $x_n=\phi_n(x)$ and $\Gamma_n^- : x_0=-f(x)$, $x_1=-\phi_1(x)$, $\cdots$, $x_n=-\phi_n(x)$. Let $K_n(f, \phi_1, \cdots, \phi_n)$ be the smallest convex body containing Γ_n^+ and Γ_n^-. The origin O surely lies in $K_n(f, \phi_1, \cdots, \phi_n)$. Let the line joining O with the point $(1, 0, \cdots, 0)$ cut $K_n(f, \phi_1, \cdots, \phi_n)$ in a point A_n. The proof of the theorem is based on the following lemma: There will exist linear combinations of the functions $\{\phi_i(x)\}$ converging uniformly toward $f(x)$ in the interval $a \leqq x \leqq b$ if, and only if, $\lim_{n\to\infty}\overline{OA}_n = 0$. The lemma is obtained by altogether elementary geometric considerations, and the theorem of Riesz follows almost immediately from it. (Received February 7, 1931.)

116. Dr. W. Cauer: *An interpolation problem of functions with positive real part.*

The design of symmetrical electric wave filters which have nearly zero "attenuation" in certain frequency intervals corresponds essentially to the solution of the following interpolation problem: $z_1 = f_1(\lambda)$ and $z_2 = f_2(\lambda)$ are rational functions of λ which are regular in the right half plane with positive real part and real on the real axis; find z_1 and z_2 so that $\log[(1+z_1)(1+z_2)/(z_2-z_1)]$ is approximately zero in certain given intervals of the positive imaginary axis and ∞ in the complementary intervals. For this it is necessary and sufficient that z_1 and z_2 be (nearly) pure imaginary on the imaginary axis, $z_1 z_2$ be approximately 1 in the first named intervals, and z_1/z_2 be approximately 1 in the complementary intervals. Both these reduced interpolation problems are independent and of the same kind. Several solutions are given. One shows the limiting case where the degree of the functions z_1 and z_2 goes to infinity and the approximation becomes ideal. Especially valuable is an approximation in the sense of Tchebycheff which is related to the transformation of elliptic integrals. The Tchebycheff parameters are calculated explicitly for all important types of filters. (Received February 7, 1931.)

117. Professor J. L. Walsh: *On interpolation and approximation by rational functions with preassigned poles.*

If C is an arbitrary limited *closed* Jordan region containing no limit point of the given set α_{in}, $i=1, 2, \cdots, n$; $n=1, 2, \cdots$, and if $f(z)$ is an arbitrary function analytic interior to C, and continuous in C, then there exist rational functions $r_n(z) = (a_{0n}z^n + a_{1n}z^{n-1} + \cdots + a_{nn})/[(z-\alpha_{1n})(z-a_{2n})\cdots(z-\alpha_{nn})]$, $n=0, 1, 2, \cdots$, approaching $f(z)$ uniformly in C. In particular, if C is the circle $|z| \leqq 1$, if the points α_{in} have no limit point interior to $|z| = A > 1$, and if $f(z)$ is analytic for $|z| < T > 1$, then the sequence of functions $r_n(z)$ of *best* approximation to $f(z)$ on C (this sequence exists and is unique) converges to $f(z)$ uniformly for $|z| < R < (A^2T+T+2A)/(2AT+A^2+1)$. This result holds whether *best approximation* is measured in the sense of Tchebycheff, in the sense of least weighted pth powers ($p>1$) on the circumference, or in the sense of least weighted pth powers ($p>1$) over the area C. There are analogous results for sequences $r_n(z)$ obtained by interpolation. (Received February 2, 1931.)

118. Dr. L. S. Kennison: *Reflections in function space.*

J. Delsarte has announced the following theorem: The necessary and sufficient condition that a kernel be a kernel of rotation (as defined by Delsarte) is that it be the resolvent of a skew-symmetric kernel with parameter 1/2. An example is given which shows that the condition is not necessary. The theorem is corrected and generalized. (Received January 19, 1931.)

119. Professor J. F. Ritt: *Quotients of differential forms.*

Let $A(u)$ and $B(u)$ be two polynomials in an arbitrary analytic function u, and its derivatives, with analytic coefficients. Let $y=A(u)/B(u)$. The question arises as to how y is to be determined if u makes both A and B vanish. What one may do, following the theory of differential equations recently developed by the author in the Transactions of this Society, is to take all functions y which, with the given u, give a solution of $By-A=0$, belonging to the *general solution* of that equation. This point of view leads to specific results of some interest. Let $F(u)$ and $G(u)$ be two rational combinations of u and its derivatives, with analytic coefficients. It is proved that there exists a third such rational combination, $H(u)$, such that H is rational in F, G and their derivatives, while F and G are each rational in H and its derivatives. This is analogous to an algebraic theorem of Lüroth. (Received February 10, 1931.)

120. Mr. W. O. Pennell: *Representation by Fourier series of functions whose successive intervals lie in separate planes differing by a phase angle.*

Suppose $\chi(x)$ is a function defined in the interval $0<x<a$ and satisfying the necessary conditions for representation in this interval by a classical Fourier series. This paper describes a quasi-Fourier series $S(x)$ in space representing the following function: $S(x)=[b^n\chi(x-na)]_{n\psi}$ in the intervals $na<x<(n+1)a$, where a, b, and ψ are any real constants and n takes the values $0, \pm 1, \pm 2, \pm 3, \cdots$ corresponding to the above intervals and the subponent notation $n\psi$ denotes the angles which the planes passing through $\chi(x-na)$ and the X axis for each interval make with the XY plane. An illustrative example is given with models showing the approximation curves twisting through space about the final form of the function in question. (Received January 24, 1931.)

121. Mr. L. E. Widmark: *On repeat operations.*

The notation $R(f(x))$ expresses a continued operation of a function upon itself. It is shown that this conception leads to a short and perspicuous notation for many at present unwieldy expressions. It also leads to a convergency solution of numerical equations which is believed new and of practical value. (Received February 6, 1931.)

122. Dr. T. C. Benton: *On continuous curves which are homogeneous except for a countable infinity of points.*

The problem of extending the results of the author's papers *On continuous curves except for a finite number of points* (Fundamental Mathematicae, vols. 13, 15) to obtain a classification of such curves with a countable infinity of non-homogeneous points gives the result that in any bounded curve of the required type the set of non-homogeneous points has a derived set which consists of a finite number (n) of points. If this number (n) is greater than one, all such curves can be constructed by taking a curve with n non-homogeneous points and replacing each arc of it by a simple type of curve homogeneous except for a countable infinity of points. If $n=1$ the problem is much more complicated. Only certain cases of it have been worked out. (Received February 7, 1931.)

123. Dr. A. A. Albert: *A classification of all pure Riemann matrices of genus $p \leqq 8$.*

Pure Riemann matrices of genus $p=2$, 3 have been classified by several authors. Their classifications are all in terms of the invariant indices of the matrices, not the far more fundamental structure of the associated multiplication algebras which also gives these indices. As a consequence the methods used failed of extension to the case $p=4$. In the present paper the author applies his recent theory of the structure of a pure Riemann matrix with a given algebra of multiplications to give a complete classification of pure Riemann matrices of genus $p \leqq 8$. In fact, a classification is given for $p=8$, $p=rp_1$, where $r=1, 2, 4,$and p_1 is unity or an odd prime. Moreover a study is made of all Riemann matrices whose multiplication algebras are those algebras occurring in the above special cases, general existence theorems being proved not merely for the case of ordinary Riemann matrices but for Riemann matrices over any real algebraic field. These algebras are algebraic fields, and three types of generalized quaternion division algebras. (Received January 28, 1931.)

124. Professor R. M. Garver: *On the Bring-Jerrard quintic.*

Most of the proofs that the general quintic equation can be reduced to the form $x^5+ax+b=0$ make no attempt to determine a and b explicitly, while those that do are very long. (See, for example, Rahts, Mathematische Annalen, vol. 28 (1886), pp. 34–60.) The purpose of the present paper is to devise a transformation, or series of transformations, which will allow a and b in the transformed equation to be computed fairly conveniently. (Received February 6, 1931.)

125. Professor R. M. Garver: *Tschirnhaus transformations and the real roots of an equation.*

Considerable use has been made of Tschirnhaus transformations in the determination of the number of real roots of equations. (See Fricke, *Algebra*, vol. 1, pp. 189 ff. and pp. 217 ff.) These applications are not, however, of an elementary nature. This paper shows how to obtain the desired criteria for quartic and quintic equations with the aid of only simple transformations and symmetric functions. (Received February 6, 1931.)

126. Dr. E. H. Cutler: *On the curvatures of a curve in Riemann space.*

An attempt is made to extend to Riemann spaces three properties of a curve in ordinary space, namely (1) the formula for the principal part of the infinitesimal distance of a point of the curve from the k-dimensional osculating geodesic space at a nearby point, (2) the equality of the first $(k-1)$ curvatures of the curve and those of its projection on the k-dimensional osculating geodesic space, and (3) the fact that if the kth curvature vanishes identically the curve lies in the k-dimensional osculating geodesic space. None of these properties holds in general, but all hold if the osculating geodesic space in question is totally geodesic, and (1) and (2) hold for the first and second curvatures. (Received February 4, 1931.)

127. Professor J. W. Lasley, Jr.: *Penosculating conics of a plane curve.*

This paper presents the results of studying certain penosculating conics associated with a given plane curve at an ordinary point of it. Constructions for these conics are obtained, and additional properties and relationships pointed out. The auxilliary parabola of Transon and the ellipse of minimum eccentricity of Wilczynski are objects of especial study. Certain results already known are extended. An involution of conics comprising the entire system of penosculating conics is introduced, generalizing results due to Wilczynski. (Received January 26, 1931.)

128. Dr. S. B. Littauer (National Research Fellow): *On convex arcs and surfaces.*

The area of a convex surface in three-space admits an intrinsic definition analogous to that for arc length, that is, as the upper bound of the areas of all the convex polyhedra inscribed in the surface. Since for two convex regions R and R' such that R is a subset of R', the boundary of R has a smaller area than that of R', the definition is equivalent to Lebesgue's within its limits of application. Arc length, it is well known, is a lower semicontinuous functional. Restricted to convex arcs, length is continuous; and with respect to the whole system of convex arcs of length greater than 0 any finite cube is compact. (Received February 2, 1931.)

129. Professor R. G. Putnam: *On spaces* V_ω.

In this paper complete spaces V_ω are defined and two theorems similar to Theorems 1 and 2 of a paper by Kuratowski (Fundamenta Mathematicae, vol. 15, p. 301) on complete metric spaces are shown to be valid in complete spaces V_ω. Theorems 3 and 4 of the same paper are extended to certain spaces V_ω. Four theorems shown by the author (this Bulletin, 1930, p. 127) to be true in spaces E are extended to spaces V_ω. (Received January 30, 1931.)

130. Mr. Hassler Whitney: *On homeomorphic graphs and the connectivity of graphs.*

Definitions will be found in a note, *Non-separable and planar graphs*, in the Proceedings of the National Academy of Sciences, February, 1931. Let G and G' be two graphs containing no 1- or 2-circuits, and let there be a (1, 1) correspondence between their arcs. Then G and G' are homeomorphic if (1) neither of the graphs is of the form ab, ac, ad. and two arcs with a common vertex in one correspond to two arcs with a common vertex in the other, or (2) the graphs are triply connected, and circuits correspond to circuits, or (3) the graphs are triply connected, and subgraphs of nullity 1 correspond to subgraphs of nullity 1. A graph is n-tuply connected if it cannot be disconnected by dropping out fewer than n vertices and their arcs. If a graph is n-tuply connected, any two of its vertices are joined by n distinct chains, and conversely. Applications to dual graphs are given: dual graphs have the same connectivity; a triply connected planar graph has a unique dual. (Received February 6, 1931.)

131. Professor Edward Kasner: *Invariance under contact transformation.*

The simplest type of differential equation invariant under all contact transformations is

$$y''' = ay''^3 + by''^2 + cy'' + d.$$

This plays the same rôle as the familiar type $y'' = Ay'^3 + By'^2 + Cy' + D$, which is of first *rank* in the author's classification for the group of all point transformations given in the American Journal of Mathematics, 1906. Besides families of ∞^3 curves, we may construct families of $2\infty^2$ curves which are contact-invariant. For example, all families obtained by applying contact transformation to the ∞^2 straight lines and the ∞^2 point-unions sonstitute such a type. We may also construct families of $\infty^2 + 3\infty^1$ curves. Such invariant family types may be characterized geometrically by a generalization of Blaschke's hexagonal web (which itself is of course only topologically invariant). If we are given $3\ \infty^2$ curves (that is, such that for any lineal element there are three curves) we may construct certain related families of *anharmonic-curvature trajectories*. This gives in all ∞^3 curves, and forms a special case of the cubic type described in the first sentence. This sub-type is contact-invariant. (Received January 31, 1931.)

132. Dr. E. H. Cutler: *On the Frenet formulas for a general subspace of a Riemann space.*

Starting with a general subspace of a Riemann space, it is shown that, by a proper choice of systems of normals and by a corresponding definition of complete convariant differentiation, formulas analogous to the Frenet formulas for a curve will be obtained. The quantities occurring there will satisfy the Gauss, Codazzi, Ricci equations, and will transform as tensors under a change of coordinate system in the subspace or under a rotation of the reference normals of any system among themselves. An extension of Meusnier's theorem is developed, and certain theorems about curves are extended to the general subspace. (Received February 4, 1931.)

133. Dr. A. A. Albert: *The integers represented by sets of positive ternary quadratic forms.*

Almost no general theorems are known on explicitly what integers are represented by positive ternary quadratic forms. In fact, as L. E. Dickson has indicated (Annals of Mathematics, vol. 28), most of these forms are *irregular*. The author studies here the problem of determining all integers represented by the set $\Sigma(d)$ of all positive classical ternaries of the same determinant d. The problem is completely solved, and it is shown that if we write $d=\gamma^2\delta$, where δ has no square factor, then $\Sigma(d)$ represents all positive integers a not of the form $a=\lambda^2\mu$, where μ has no square factor, $\mu=\alpha\delta$, α is prime to d, $\alpha\equiv 7 \pmod 8$, and the Jacobi symbol $(f|\alpha)=1$ for every factor f of d. This result is easily shown to imply that every set $\Sigma(d)$ is regular in the Dickson sense and that no such set represents all positive integers. Also every set $\Sigma(n, d)$, $n>3$, of all positive classical n-ary quadratic forms of determinant d represents all positive integers. (Received January 28, 1931.)

134. Dr. Frances Harshbarger: *The geometric configuration defined by a special algebraic relation of genus* 4.

This paper discusses the curve defined by the special form $t_1^3\tau_1^2\tau_2+t_1^2t_2\tau_2^3+t_1t_2^2\tau_1^3-t_2^3\tau_1\tau_2^2=0$, where t_1/t_2 and τ_1/τ_2 define the generators of a quadric surface. This form was first investigated by Gordan (Mathematische Annalen, vol. 13 (1878)) in connection with the solution of the quintic equation. (Received February 14, 1931.)

135. Professors C. L. E. Moore and Philip Franklin: *Geodesics of Pfaffians.*

In a Pfaffian, there is a class of curves whose principal normals are normal to the Pfaffian. These have been called "geodesics," although their dimensionality is one lower than that of pairs of points in the Pfaffian. We here study the lines of shortest length joining pairs of points, obtaining their differential equations, their finite form for some simple examples, and some of their local properties. The earlier class are only curves of shortest length for integrable Pfaffians. (Received February 11, 1931.)

136. Professors C. L. E. Moore and Philip Franklin: *Pfaffians in parametric form.*

In this paper we show that the method of parametric representation for surfaces, which has been assumed by some writers to be applicable to Pfaffians, only applies to certain very restricted classes of Pfaffians. These classes are determined, as well as those types of Pfaffian in three-space for which the arc element can be represented in terms of two parameters. (Received February 11, 1931.)

137. Professor Philip Franklin: *Regions of positive and negative curvature on closed surfaces.*

The question as to topological restrictions on regions of positive and negative curvature has been raised by J. Douglas (this Bulletin, vol. 36, p. 798). In this note we show by examples that the conjectured properties do not hold for general closed surfaces. If, however, we impose the restriction that the points on a surface of genus one at which the curvature is zero form simple curves, and include no umbilics or points at which the normal section to the surface has a point of inflection, the regions in question are not shrinkable to a point. (Received February 11, 1931.)

138. Professor Philip Franklin: *Two functional equations with integral arguments.*

E. T. Bell (American Mathematical Monthly, vol. 37, p. 484) has formulated two functional equations with integral arguments whose general solution is of importance in a question on types of arithmetic. We here solve them and show that the general solution of the first equation involves a single function of the variable, while that of the second involves an enumerable number of such functions. (Received February 11, 1931.)

139. Professor Philip Franklin: *The geometric interpretation of some elementary formulas of analytic geometry.*

The condition that three points lie on a line is an expression of the fact that the area of the triangle with these points as vertices vanishes. The connection of the condition that three lines meet in a point with the vanishing of the triangle having these lines as sides is not so direct, and has not been explicitly brought out. In this note we show that when the equations are suitably normalized, the condition admits of geometric interpretation. The generalization to three and n dimensions is indicated. (Received February 11, 1931.)

140. Miss Ruth W. Stokes: *A geometric theory of solution of linear inequalities.*

The author developes a theory of systems $\xi^i \geqq 0$, where the ξ's are linear homogeneous expressions in n unknowns. Each condition is represented in n-dimensional euclidean space by the point which has for coordinates the known coefficients of the corresponding form ξ. A solution is accordingly an oriented $(n-1)$-flat which does not separate any pair of the points representing the given conditions. The paper determines the general solution of the following problem: find a solution $\xi \geqq 0$ which contains all points in a subset A of the given representative points and which contains no point in a second subset B. A general solution is thus obtained for any system composed of a finite number of conditions from each of the three types (1) $\xi \geqq 0$, (2) $\xi > 0$, (3) $\xi = 0$. Such a system includes as special cases the system composed solely of conditions of type (1) studied by Minkowski (*Geometrie der Zahlen*, pp. 40–45); that composed solely of conditions of type (2) studied by Dines (Annals of Mathematics, vol. 20, pp. 191–199); and that composed solely of conditions of type (3), which is a system of linear homogeneous equations. The method is geometric throughout. (Received March 4, 1931.)

141. Mr. C. C. Torrance: *On plane Cremona triadic characteristics.*

In this paper is developed a new procedure for obtaining the characteristics of all plane Cremona transformations. This procedure is perfectly regular and can be simply and explicitly formulated. Furthermore it develops the characteristics in such a way that all the characteristics obtained by any given number of steps are included in one single general formula. This formula includes only geometric characteristics. The latter part of the paper is devoted to the beginning of an exhaustive investigation of the ways in which the parameters introduced may be evaluated so that triadic characteristics result. Twelve infinite sequences of two-parameter triadics are obtained and their conjugates determined. All of these triadics have for one of their groups a single point of highest multiplicity. (Received March 4, 1931.)

142. Professor C. C. MacDuffee: *A method for determining the canonical basis of an ideal in an algebraic field.*

While the existence of a basis for every ideal in an algebraic field is easily proved, an effective method for calculating it has been developed only in the case of quadratic ideals (M. Cipolla, Atti Academie Gionia Scientica Nationale in Catonia (5), vol. 10 (1917), No. 20), and the method used is too complicated to invite attempts to generalize it. It is here shown that a basis for the ideal of $(a_1, a_2, \cdots, a_k)$ can easily be obtained in matric form as the greatest common right divisor of certain integral matrices obtained directly from the a's. The process is entirely practicable. (Received January 28, 1931.)

143. Dr. E. J. McShane, National Research Fellow: *On the necessary condition of Weierstrass in the multiple integral problem of the calculus of variations.*

The principal result arrived at in this paper is the following: Given a class K of functions $z(x, \cdots, w)$ for which the point $(x, \cdots, w)$ lies in a set D of n-space and $(z, x, \cdots, w)$ in a set A of $n+1$-space, and an integral $F=\int_D f(x, \cdots, w, z, z_x, \cdots, z_w)dx \cdots dw$. If the function $\zeta(x)$ minimizes F in K, and L is the set of points $(x, \cdots, w)$ interior to D for which $(\zeta(x, \cdots, w), x, \cdots, w)$ is interior to A and of indifference with respect to K and A, and near which ζ satisfies a Lipschitz condition, then for almost all points of L the Weierstrass function $E(x, \cdots, w, \zeta, \zeta_x, \cdots, \zeta_w, z_x, \cdots z_w)$ is non-negative for all sets of numbers $z_x, \cdots, z_w$. If f satisfies a certain supplementary condition, we can remove the restriction that ζ satisfy a Lipschitz condition near $(x, \cdots, w)$. (Received January 27, 1931.)

144. Professor Ernest P. Lane: *Conjugate nets and the lines of curvature.*

The purpose of this paper is twofold; first, to make some contributions to the profective differential geometry of an arbitrary conjugate net on a general analytic surface in ordinary space; and second, to connect this geometry with the metric differential geometry of the lines of curvature. Two quadrics called *conjugate osculating quadrics* are associated with each point of a curve on the surface sustaining a given conjugate net. These are analogous to the *asymptotic osculating quadrics* of Bompiani and Klobouček, the difference being that three consecutive tangents of one family of the given conjugate net are constructed at a point of the curve to determine a quadric, instead of three consecutive asymptotic tangents. The transition to metric geometry is affected by a transformation which makes it possible to study metrically for the lines of curvature all configurations defined in the projective theory for a general conjugate net in ordinary space. (Received February 4, 1931.)

145. Mr. E. S. Wilks: *On the distribution of statistics in samples from a normal universe of two variates with matched sampling of one variable.*

A type of sampling frequently used in statistical investigations may be formulated abstractly by considering a sample of s items drawn from a normally distributed universe of two variables x, y in such a way that the distribution of the x's is made identical with a given distribution $D(x)$, independently of the y's. This paper deals with the derivation of the distribution functions of the important statistics pertaining to the y's for sampling under such restrictions. The method used may be described in a general way as an extension of Romanovsky's integral equation method of finding distributions of statistics from normal universes for strictly random sampling. Expressions were also found for the moments, expected values and standard errors of these statistics, some of which were obtained in the form of confluent hypergeometric series which were transformed into power series in $1/s$ by a relation due to Kummer, thus enabling one to make convenient approximations. (Received February 20, 1931.)

146. Dr. J. J. Gergen: *Note on a theorem of Bôcher and Koebe.*

In this paper the following theorem is established:

Theorem I. If $v(x, y)$ is positive and harmonic in a plane region R, if $u(x, y)$ is continuous with its first partial derivatives in R, and if, for every circle C contained in R, $\int_C v\partial u/\partial n\, ds = \int_C u\partial v/\partial n\, ds$, where n is the exterior normal to C, then u is harmonic in R. Taking $v=1$ here, a well-known theorem, discovered independently by Bôcher and Koebe, is obtained. The analog of Theorem I holds in space. (Received February 24, 1931.)

147. Dr. J. Yerushalmy, National Research Fellow: *On the representative space S_9 of the plane cubics.*

The plane cubics are mapped on the points of an S_9. Cubics with a given singularity on the points of a spread in S_9. The orders of the various spreads are determined as well as their multiplicities with respect to each other, and the linear spaces by which they are ruled. A more detailed study is made of the surface $F_2{}^9$ of cubics degenerating into a triple line determining the curves on it, its tangentsplanes and hyperplanes, its projection on S_3, and its map on the plane. Most of the results have been extended to the general representative space $S_N(N=n(n+3)/2)$ of the curves of order n in the plane. (Received March 1, 1931.)

148. Professor H. T. Davies: *Properties of the operator $z^{-v}\log z$, where $z=d/dx$.*

The operator discussed by this paper is derived from the fractional operator z^{-v} by differentiation with respect to v. Its inverse is obtained by means of a proper integration from 0 to ∞ and various generalizations are suggested. The paper shows that the generalized Leibnitz formula for the expansion of the operation $F(x, d/dx)uv$ holds when $F(x, z)$ is the logarithmic operator and that in consequence the Bourlet operational product is also valid when logarithmic terms are included in either factor. Commutation rules for the operator are derived and application is finally made to the solution of several types of integral equations. (Received March 2, 1931.)

149. Professor H. S. Wall: *Convergence criteria for continued fractions.*

A certain transformation $b=[a]$ takes the continued fraction $F(a, z)=1/a_1z+1/a_2+1/a_3z+\cdots$, into $F(b, z)$, (a_i, b_i real and not zero). If $\sum|b_i|$ converges $F(a, z)$ converges if and only if $\lim_n|g_n{}^a|=\infty$, where $g_n{}^a+a_2+a_4+\cdots$

$+a_{2n}$, $(g_0^a=0)$. Let $c=[b]$ take $F(b, z)$ into $F(c, z)$. Then if $\sum |c_i|$ converges $F(a, z)$ converges if and only if (*i*)$\sum c_{2i+1}=0$ or (ii) $\sum c_{2i+1}\neq 0$, $\sum b_{2i+1}=0$. When $F(a, z)$ converges the limit is a meromorphic function of z. Explicit formulas for the b_i, c_i are

$$b_{2i+1} = b_1^2 a_{2i}/(1 - b_1 g_1^a)(1 - b_1 g_{i-1}^a), \quad b_{2i+2} = a_{2i+1}(1 - b_1 g_i^a)^2/b_1^2,$$
$$c_{2i+1} = c_1^2 a_{2i-1}(1 - b_1 g_{i-1}^2)^2/b_1^2(1 - c_1 g_i^b)(1 - g_{i-1}^b),$$
$$c_{2i+2} = b_1^2 a_{2i}(1 - c_1 g_i^b)^2/c_1^2(1 - b_1 g_i^a)(1 - b_1 g_{i-1}^a).$$

(Received March 3, 1931.)

150. Professor John Eiesland: *On the ruled $V_4^{(4)}$ in S_5 associated with a Schlafli hexad of 3-flats.*

The present paper gives an account of the singular loci of a ruled $V_4^{(4)}$ in S_5, the general equation of which has been deduced by C. R. Rupp, (Transactions of this Society, vol. 31, p. 587). By a projective transformation the equation of the surface has been reduced to a second order determinant equated to zero, as was done by the author for the generic case in a paper appearing in the Rendiconti. The surface has *27* double-lines of which 15 are denoted as regular double-lines, 5 in each of the six flats of the hexad, and 12 accessory double-lines, two in each flat which intersect the five regular double-lines belonging to the flat. These accessory double-lines are tac-loci on the $V_4^{(4)}$. The necessary and sufficient conditions that the surface shall be self-dual are also found. This surface has six nodal quadrics, one in each flat. (Received March 4, 1931.)

151. Professor L. M. Graves: *On an existence theorem of the calculus of variation.*

Hahn has given a rather general existence theorem for the minimum of a semi-definite integral in the plane. (See Wiener Berichte, vol. 134 (1925) pp. 437–447.) In making the proof he has recourse to a geometric form of argument which is not extensible to integrals in space of three or more dimensions. To fill this gap, the following lemma, which holds for any number of dimensions, is proved. Let the sequence of rectifiable curves C_n converge to a single point X_0, and suppose that the Weierstrass function $E(x_0, x', X')\geqq 0$ for all directions x' and X', while $E(x, x', -x')>0$ for all directions x'. Then there exists a positive number m such that lim sup $(JC_n)\geqq m$ lim sup $L(C_n)$, lim inf $J(C_n)$ $\geqq m$ lim inf $L(C_n)$. Here $J(C)$ denotes the value of the integral $\int F(x, x')dt$ along the curve C, while $L(C)$ denotes the length of C. For the case of two dimensions, a much simpler proof than Hahn's is indicated. (Received March 4, 1931.)

152. Mr. J. M. Feld: *Analytic curve for which the chord equals the arc.*

A proof is given that curves in three-space, such that the chord OP equals the arc OP for every point P, lie in minimal planes. This result was announced by E. Kasner at the thirty-first annual meeting of the Society. An investigation of curves having the same property in four-space is also made. (Received March 4, 1931).

153. Mr. J. M. Feld: *The generalized pedal transformation.*

If on the tangents t to a plane curve K points P are found such that the cross-ratio of PA, PB, PC and t be constant, A, B, and C being three fixed non-collinear points, the locus of P is defined as the generalized pedal of K with respect to A, B, C. If B, and C are the circle points at infinity then AP will make a constant angle with t, the magnitude of which is determined by the fixed cross-ratio. The generalized pedal transformation is proved to be a contact transformation. It is shown that W-curves are their own generalized pedals and a number of properties of the transformation are determined. (Received March 4, 1931.)

154. Mr. J. M. Feld: *Birational contact transformations.*

A proof is given that a contact transformation in the plane is birational if and only if its directrix equation has the following property: that if either set of its variables be regarded as parameters of the directrix equation represent a homoloidal family of curves. Birational contact transformations are shown to be equivalent to a sequence of transformations, the first a Cremona transformation, then a polarity and finally, another Cremona transformation. Homogeneous coordinates are used throughout. (Received March 4, 1931.)

155. Professor B. A. Bernstein: *On unit-zero Boolean representations of operations and relations.*

In a previous paper the author (in cooperation with Mr. N. Debely) made basic notion of *unit-zero functions* in a method of obtaining arithmetic representations of arbitrary operations and relations in a finite class of elements. The present paper determines to what extent this method can be used analogously to obtain *Boolean* representations of arbitrary operations and relations. (Received February 11, 1931.)

156. Professor B. A. Bernstein: *Note on the condition that a Boolean equation have a unique solution.*

Whitehead has obtained certain results concerning the condition that a Boolean equation have a unique solution. Professor Bernstein obtains these results much more simply than does Professor Whitehead, and he also points out the geometry underlying them. (Received February 11, 1931.)

157. Dr. Leonard Carlitz: *A problem in additive arithmetic.*

An asymptotic expression is found for $\sum\alpha(n_1)\alpha(n_2)\cdots\alpha(n_\nu)$, the summation extending over all the partitions of a large integer n into ν parts ($\nu\geqq 3$), $\alpha(n)$ satisfying certain conditions. The method is an application of the Hardy-Littlewood analysis, particularly as applied by them to the Goldbach Problem. The paper will appear in the Quarterly Journal of Mathematics. (Received March 2, 1931.)

158. Dr. Leonard Carlitz: *A class of algebraic fields of characteristic p.*

The binomial equation $x^m = M$ over the field of rational functions with coefficients in a Galois field of order p^n, where m is a divisor of p^n-1, defines a field resembling in many particulars the ordinary circular fields. Its properties are here developed in a very simple way; this is especially true of the theory of units. (Received March 2, 1931.)

159. Dr. Leonard Carlitz: *Quadratic fields over the field of rational functions, modulo* 2.

Following Artin (Mathematische Zeitschrift, vol. 19 (1924), pp. 154–246), who limited himself to the case of an odd prime modulus, a treatment of the elementary and analytic ideal-theory of the fields described is given. An interesting feature is that the number of fields having the same discriminant is infinite. (Received March 2, 1931.)

160. Dr. Leonard Carlitz: *New Diophantine automorphisms.*

Applying a simple principle, it is indicated how an unlimited number of Diophantine automorphisms may be derived from certain invariants of binary forms of odd order. Eisenstein's quaternary quartic appears as a special case. (Received March 2, 1931.)

161. Professor Morgan Ward: *Systems of Appell polynomials.*

The Appell polynomials are generalized by considering sets of M polynomials $P_r^{(N)}(x)$, $(r=1, \cdots, N)$ of degrees $N=0, 1, 2, \cdots$, satisfying the relations

$$\frac{dP_r^{(N)}(x)}{dx} = \sum_{j=1}^{M} d_{rj}P_j^{(N-1)}(x), \qquad (N = 1, 2, \cdots),$$

where the d_{rj} are constants independent of N and x. The formal algebraic theory of the Appell polynomials is shown to carry over to these sets of polynomials with only slight modifications. (Received March 6, 1931.)

162. Professor Morgan Ward: *Some arithmetical properties of sequences satisfying a linear recursion relation.*

The author gives a number of congruences to a prime modulus satisfied by solutions of a linear difference equation with constant coefficients when its characteristic equation is irreducible with respect to the modulus. (Received March 6, 1931.)

163. Professor Morgan Ward: *Conditions for the solubility of the diophantine equation.*

In this paper the author obtains conditions that the diophantine equation $x^2-N^2Dy^2=-1$ be soluble not depending on continued fraction theory. It is sufficient to consider the equation when N is an odd prime P and D is squarefree and prime to P. If D is a non-residue of P, a necessary and sufficient condition that $x^2-P^2Dy^2=-1$ be soluble is that $x^2-Dy^2=-1$ be soluble. If D is a residue of P, this condition though necessary is not sufficient for solubility. A second necessary condition is stated in terms of the quadratic character of the number $u+iv$ with respect to the number $a+ib$ in the field $F(i=\sqrt{-1})$, where (u, v) is the least solution of $x^2-Dy^2=-1$ and the norm of $a+bi$ is P.

In case P is of the form $8n+5$, these two conditions are sufficient for solubility. Application is made to the equation $x^2-5P^2y^2=-1$. (Received March 6, 1931.)

164. Professor Morgan Ward: *Orthogonal and periodic systems of Appell Polynomials.*

The work of the previous paper on Systems of Appell polynomials is extended by subjecting the polynomials to additional restrictions which generalize the orthogonal and periodic properties of the ordinary Appell polynomials. In particular, the periodic properties of Appell polynomials recently developed by the author (Annals, vol. 31 (1930) pp. 43–51) are extensively generalized. (Received March 6, 1931.)

165. Professor Morgan Ward: *Orthogonal Appell polynomials.*

It is shown that under certain restrictions as to convergence, the only Appell polynomials which are orthogonal for some range are substantially the Hermite polynomials. (Received March 6, 1931.)

166. Professor Morgan Ward: *The linear form of numbers represented by a homogeneous polynomial in any number of variables.*

Necessary conditions are obtained that all of the numbers properly represented by a homogeneous polynomial in any number of variables may be of the linear forms nz, $nz+a_1, \cdots$ or $nz+a_r$ where n is any integer of $a_1, \cdots, a_r$ are r distinct integers less than n and prime to it. Application is made to the problem of obtaining binary forms such that the prime factors of all numbers properly represented by the forms are either of the type $nz\pm 1$ or divisors of n. (Received March 6, 1931.)

167. Professor W. M. Whyburn: *On the Lebesgue integral.*

It is shown that a necessary and sufficient condition for a bounded function $f(x)$ on $X: a\leqq x\leqq b$ to be measurable on X is that there exists a sequence of *simple functions* (*horizontal* or *step functions*) which approaches $f(x)$ almost everywhere on X. It follows from this that the treatment given by Friedrich Riesz in Acta Mathematica, Volume 42, pp. 191–205, yields an integral which is identical with the Lebesgue integral. (Received March 3, 1931.)

168. Dr. Leo Zippin: *Generalization of a theorem due to C. M. Cleveland.*

At the suggestion of Professor R. L. Moore a plane theorem recently announced by Cleveland (Abstract 36-9-331) is generalized to euclidean spaces of dimension n greater than 2. It is proven that if C is a continuous curve non-dense in E_n, $n>2$, and B is a closed and totally disconnected subset of E_n such that $B''=B\cdot C$ does not separate (locally separate) C, then there exists a ray L in E_n such that 1) $L>B$, 2) $L\cdot C=B^0$ is totally disconnected, 3) B^0 does not separate (locally separate) C, 4) $B^0-B<C^{n-1}$, where C^{n-1} denotes the points of C which are of dimension at least $n-1$ in C, 5) the endpoint of L is an arbi-

trary point of B, 6) if B is bounded, L is an arc with arbitrary endpoints in B. It is proven also that on the hypothesis above, there exists an acyclic continuous curve T of E_n with properties analogous to those of L and such that B is the set of endpoints of T. Certain results of Moore and Zarankiewicz relating to paths that do not separate a given continuous curve, and to accessibility of points of one dimensional continua in 3-space, respectively, are given considerable generalization. (Received February 12, 1931.)

169. Professor J. H. Roberts: *A property related to completeness.*

In December, 1926, R. L. Moore presented an axiom 1′ (see abstract No. 23, this Bulletin, vol. 33, p. 141) which gives a property resembling completeness. However, Moore has given an example of a space satisfying his axiom 1′ yet which is not metric and, therefore, not complete. He raised this question: *Is every metric space satisfying Axiom 1′ a complete metric space?* The present paper answers this question in the affirmative. (Received March 14, 1931.)

170. Dr. D. H. Lehmer: *Arithmetic periodicity of Bessel functions.*

The object of this paper is to study the arithmetical properties of the solutions of the difference equation $U_{n+1}=(an+b)U_n+cU_{n-1}$, where a, b, and c are integers. This equation has for special cases the equations $U_{n+1}=nU_n$ and $U_{n+1}=bU_n+cU_{n-1}$ on which the topics allied to Wilson's theorem and Lucas' functions are respectively founded. The complete solution of the given difference equation is given in terms of Bessel's functions and most simply by Lommel's polynomials. The chief aim of the investigation is to determine the period of the general solution U_n taken with respect to a modulus M, any number prime to a. The principal theorem is as follows: If ac is prime to M and if e is the exponent to w which c belongs modulo M, then the proper period of U_n (mod M) is the product of the L.C.M. of e and M by some divisor of twice their G.C.D. Examples show that these limits for the period cannot be made sharper. (Received March 10, 1931.)

171. Dr. D. H. Lehmer: *On the arithmetic of double series.*

This investigation is concerned with the properties of sums of the type $h(n)=\sum f(i)g(j)$, where f and g are numerical functions and where i and j range over all positive integral solutions of $\psi(i,j)=n$. The function $\psi(i,j)$ is a single and integral valued function and determines a method of grouping the terms of a double series. The cases $\psi(i,j)=i+j-1$ and ij are well known and give rise to Cauchy and Dirichlet multiplication. The general symbolic product $h(n)$ is called ψ-multiplication. A sytem of 5 conditions on ψ is found to be necessary and sufficient to render ψ-multiplication finite, associative, commutative, and to insure the existence of a single unit of multiplication, and unique quotient when division is possible. The existence of infinite series expansions, whose connection with ψ is the same as that of power and Dirichlet series with the special cases $\psi(i,j)=i+j-1$ and ij, is shown to depend upon the solutions of certain functional equations. Several examples and special cases are considered. (Received March 10, 1931.)