ON LOCI OF $(r-2)$-SPACES INCIDENT WITH CURVES IN r-SPACE

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Let s_t lines and t curves $C^{m_1}, C^{m_2}, \cdots, C^{m_t}$ of orders $m_1, m_2, \cdots, m_t$ and deficiencies $p_1, p_2, \cdots, p_t$, respectively, be given in general positions in r-space. In this paper, we propose to determine the number, $N_r^{(t)}$, of $(r-2)$-spaces that are incident with the s_t lines and meet C^{m_1} n_1 times, C^{m_2} n_2 times, $\cdots$, C^{m_t} n_t times, where

$$s_t + n_1 + n_2 + \cdots + n_t = 2r - 2, \tag{A}$$

and to deduce a few consequences from the formula for this number. The formula which we shall derive is obviously a function of r, n_i, m_i, and p_i, $(i=1, 2, \cdots, t)$. The derivation of this formula can be accomplished algebraically,* or by Schubert's symbolic calculus,† by the functional method,‡ or by the method of decomposition.‡ In the present work we find it convenient to adopt the method last named as it yields the desired result with the least difficulty.

A curve C^m may be decomposed in various ways into component curves the sum of whose orders is equal to m, but we wish to decompose it completely, that is, into m lines forming a skew polygon Γ of m sides and $Q^{(1)} = m-1+p$ vertices where p is the deficiency of C^m. The non-adjacent vertices of Γ arrange themselves in groups each consisting of a certain number, q, of members. Let $Q^{(q)}$ denote the number of such groups. As we shall have frequent use for this number, we record the following which can be easily verified:

$$Q^{(1)} = \binom{m-1}{1} + \binom{p}{1},$$

* Salmon, *Modern Algebra*, 4th ed., Lesson 19.

† Schubert, *Kalkül der Abzählenden Geometrie*, Leipzig, 1879.

‡ Severi, *Riflessioni intorno ai problemi numerativi concernenti le curve algebriche*, Rendiconti Istituto Lombardo, (2), vol. 54 (1921), pp. 243–254.

$$Q^{(2)} = \binom{m-2}{2} + \binom{m-3}{2}\binom{p}{1} + \binom{p}{2},$$

(B) . . . .

. . . .

$$Q^{(q)} = \sum_{j=0}^{q} \binom{m-q-j}{q-j}\binom{p}{j};$$

$Q^{(0)}$ is to be taken equal to unity.

Now let $t=0$. Then the symbol $N_r^{(0)}$ denotes the number of $(r-2)$-spaces incident with $s_0=2r-2$ general lines in r-space. Since the number of lines incident with $2r-2$ general $(r-2)$-spaces given in r-space is*

$$\frac{(2r-2)!}{r!(r-1)!},$$

we assume by the principle of duality, or we can prove independently, that $N_r^{(0)}$ is equal to this number; that is,

$$N_r^{(0)} = \frac{(2r-2)!}{r!(r-1)!}. \tag{1}$$

Diminishing r by w, we have

$$N_{r-w}^{(0)} = \frac{(2r-2w-2)!}{(r-w)!(r-w-1)!} \tag{1a}$$

for the number of $(r-w-2)$-spaces that meet $2r-2w-2$ general lines in $(r-w)$-space, which is also the number of $(r-2)$-spaces that pass through w general points and meet $2r-2w-2$ general lines in r-space.

We proceed now to determine, for the case $t=1$, the number $N_r^{(1)}$ of $(r-2)$-spaces that meet n^1 times a given curve C^{m_1} and are incident with $s_1=2r-2-n_1$ given lines in r-space. Replace C^{m_1} by an m_1-sided skew polygon Γ_1 with $Q_1^{(1)}=m_1-1+p_1$ vertices. Any m_1 general lines determine n_1 by n_1 with the s_1 given lines

$$\binom{m_1}{n_1} N_r^{(0)}$$

* C. Segre, *Mehrdimensionale Räume*, Encyklopädie der Mathematischen Wissenschaften, III_2, 7, p. 814. Also B. C. Wong, *On the loci of the lines incident with k $(r-2)$-spaces in S_r*, this Bulletin, vol. 34, pp. 715–717.

$(r-2)$-spaces all incident with the s_1 given lines and each incident with n_1 of the m_1 lines. But the m_1 lines or sides of Γ_1 have $Q_1^{(1)}$ incidences each on two of the lines. Through each vertex of Γ_1 pass

$$\binom{m_1-2}{n_1-2}N_{r-1}^{(0)}$$

$(r-2)$-spaces all incident with the s_1 given lines and each incident with n_1-2 of the m_1-2 sides on which the vertex does not lie. Therefore the $Q_1^{(1)}$ vertices of Γ_1 determine

$$\binom{m_1-2}{n_1-2}N_{r}^{(0)}Q_1^{(1)}$$

such $(r-2)$-spaces. As these $(r-2)$-spaces are improper n_1-uple secant $(r-2)$-spaces of the degenerate curve C^{m_1} incident with the s_1 lines, we deduct their number from

$$\binom{m_1}{n_2}N_{r}^{(0)}.$$

To the result we now add

$$\binom{m_1-4}{n_2-4}N_{r-2}^{(0)}q_1^{(1)},$$

which is the number of $(r-2)$-spaces each passing through a pair of non-adjacent vertices of Γ_1 and meeting the s_1 given lines and n_1-4 of the m_1-4 sides of Γ_1 not passing through the vertices. Continuing in this manner, we find

$$N_r^{(1)} = \sum_{q_1=0}^{h_1}(-1)^{q_1}\binom{m_1-2q_1}{n_1-2q_1}N_{r-q_1}^{(0)}Q_1^{(q_1)}, \tag{2}$$

where $h_1=n_1/2$ if n_1 is even and $h_1=(n_1-1)/2$ if n_1 is odd. Replacing r by $r-w$, we have

$$N_{r-w}^{(1)} = \sum_{q_1=0}^{h_1}(-1)^{q_1}\binom{m_1-2q_1}{n_1-2q_1}N_{r-w-q_1}^{(0)}Q_1^{(q_1)} \tag{2a}$$

as the number of $(r-2)$-spaces that pass through w given general points and meet a given curve C^{m_1} n_1 times and also meet $s_1-2w=2r-2w-2-n_1$ given lines in r-space.

Now let $t=2$. Then $s_2=2r-2-n_1-n_2$. To determine the number, $N_r^{(2)}$, of $(r-2)$-spaces that meet two curves C^{m_1}, C^{m_2} respectively n_1, n_2 times and are incident with s_2 given general lines, we decompose C^{m_2} into m_2 lines forming an m_2-sided skew polygon Γ_2 with $Q_2^{(1)}=m_2-1+p_2$ vertices. There are

$$\binom{m_2}{n_2}N_r^{(1)}$$

$(r-2)$-spaces incident with any m_2 general lines of r-spaces n_2 at a time which are also incident with the s_2 given lines. From this number we deduct

$$\binom{m_2-2}{n_2-2}N_{r-1}Q_2^{(1)},$$

which is the number of $(r-2)$-spaces all incident with the s_2 given lines and each incident with a vertex of Γ_2 and with n_2-2 of the m_2-2 sides of Γ_2 on which the vertex does not lie. Continuing as in the preceding paragraph, we find

$$N_r^{(2)} = \sum_{q_2=0}^{h_2}(-1)^{q_2}\binom{m_2-2q_2}{n_2-2q_2}N_{r-q_2}^{(1)}Q_2^{(q_2)},$$

where $h_2=n_2/2$ if n_2 is even and $h_2=(n_2-1)/2$ if n_2 is odd. Putting $w=q_2$ in (2a) and substituting the result in the above we have, after simplifying,

$$(3)\quad N_r^{(2)} = \sum_{q_1=0}^{h_1}\sum_{q_2=0}^{h_2}(-1)^{q_1+q_2}\binom{m_1-2q_1}{n_1-2q_1} \times \binom{m_2-2q_2}{n_2-2q_2}N_{r-q_1-q_2}^{(0)}Q_1^{(q_1)}Q_2^{(q_2)}.$$

To determine $N_r^{(3)}$, $N_r^{(4)}$, $\cdots$, we proceed in a similar manner. Finally, we arrive at the desired formula:

$$(4)\quad N_r^{(t)} = \sum_{q_1=0}^{h_1}\sum_{q_2=0}^{h_2}\cdots\sum_{q_t=0}^{h_t}(-1)^{q}N_{r-q}^{(0)} \times\binom{m_1-2q_1}{n_1-2q_1}\binom{m_2-2q_2}{n_2-2q_2}\cdots\binom{m_t-2q_t}{n_t-2q_t}Q_1^{(q_1)}Q_2^{(q_2)}\cdots Q_t^{(q_t)},$$

where

$$q = q_1+q_2+\cdots+q_t,$$

and

$$h_i = n_i/2, \text{ if } n_i \text{ is even,}$$

and

$$h_i = (n_i-1)/2, \text{ if } n_i \text{ is odd.}$$

Now we deduce a few consequences from this formula. For

$t=0$ and $t=1$, we have (1) and (2) respectively. If we put in(2) $n_1=2r-2$, we obtain, since $h_1=r-1$,

$$(5) \qquad N_r^{(1)} = \sum_{q_1=0}^{r-1}(-1)^{q_1}\binom{m_1-2q_1}{2r-2-2q_1}N_{r-q_1}^{(0)}Q_1^{(q_1)},$$

as the number of $(2r-2)$-secant $(r-2)$-spaces of an r-space curve C^{m_1}. For $r=2$ and $r=3$, (5) becomes respectively

$$\overline{N}_2^{(1)} = \binom{m_1}{2} - Q_1^{(1)} = \frac{1}{2}(m_1-1)(m_1-2) - p_1,$$

and

$$\overline{N}_3^{(1)} = \sum_{q_1=0}^{2}(-1)^{q_1}\binom{m_1-2q_1}{4-2q_1}N_{3-q_1}^{(0)}Q_1^{(q_1)}$$
$$= \frac{1}{12}(m_1-2)(m_1-3)^2(m_1-4) - \frac{1}{2}(m_1-3)(m_1-4)p_1$$
$$+ \frac{1}{2}p_1(p_1-1),$$

the former giving the number of double points on a plane curve C^{m_1} of deficiency p_1 and the latter giving the number of quadrisecant lines of a 3-space curve C^{m_1} of deficiency p_1.

If we put $m_1=2r-1$, $p_1=0$ in (5), we have, taking account of (B) and (1a),

$$\sum_{q_1=0}^{r-1}(-1)^{q_1}\frac{(2r-q_1-1)!}{q_1!(r-q_1)!(r-q_1-1)!},$$

which is equal to unity. That is, a rational curve C^{2r-1} of order $2r-1$ in r-space has one and only one $(2r-2)$-secant $(r-2)$-space.

Again, if we put in (2) $n_1=2r-3$ and hence $s_1=1$, $h_1=r-2$, we obtain

$$(6) \qquad \overline{N}_r^{(1)} = \sum_{q_1=0}^{r-2}(-1)^{q_1}\binom{m_1-2q_1}{2r-3-2q_1}N_{r-q_1}^{(0)}Q_1^{(q_1)}.$$

This is the number of $(2r-3)$-secant $(r-2)$-spaces of an r-space curve C^{m_1} that meet a given line, and is therefore the order of the hypersurface formed by the ∞^1 $(2r-2)$-secant $(r-2)$-spaces of C^{m_1}. For $r=2$, the formula gives m_1, that is, the locus of points on a plane curve C^{m_1} is the curve itself. For $r=3$, we have

$$\overline{\overline{N}}_3^{(1)} = \binom{m_1}{3} - \binom{m_1 - 2}{1}(m_1 - 1 + p_1)$$

$$= \frac{1}{3}(m_1 - 1)(m_1 - 2)(m_1 - 3) - (m_1 - 2)p_1,$$

for the order of the trisecant surface of a 3-space curve C^{m_1}.

It is of interest to note that the result of substituting $m_1 = 2r-2$ and $p_1 = 0$ in (6) is, if account be taken of (B) and (1a),

$$\sum_{q_1=0}^{r-2}(-1)^{q_1}\frac{(2r - q_1 - 2)!(2r - 2q_1 - 2)}{q^1!(r - q_1)!(r - q_1 - 1)!} = 2.$$

Therefore, the locus of the ∞^1 $(r-2)$-spaces that meet a rational r-space curve C^{2r-2} of order $2r-2$ is always a quadric hypersurface.

Returning to the general formula (4), we see that it is identical with (3) if $t=2$. Let $s_2=0$. Then, from (A), $n_1+n_2=2r-2$. Consider the case $n_1=n_2=r-1$. Then (3) becomes

$$(7)\quad N_r^{(2)} = \sum_{q_1=0}^{h_1}\sum_{q_2=0}^{h_2}(-1)^{q_1+q_2}\binom{m_1 - 2q_1}{r - 1 - 2q_1} \times \binom{m_2 - 2q_2}{r - 1 - 2q_2} N_{r-q_1-q_2}^{(0)} Q_1^{(q_1)} Q_2^{(q_2)},$$

where $h_1=h_2=(r-1)/2$ if r is odd and $h_1=h_2=(r-2)/2$ if r is even. This gives the number of common $(r-1)$-secant $(r-2)$ spaces of two curves C^{m_1} and C^{m_2} in r-space. The case $m_1=m_2=r$ and $p_1=p_2=0$ is worth noting. Formula (7) for this case gives

$$\sum_{q_1=0}^{h_1}\sum_{q_2=0}^{h_2}(-1)^{q_1+q_2}\frac{(r - 2q_1)(r - 2q_2)}{(r - q_1 - q_2)} \times \binom{r - q_1}{q_1}\binom{r - q_2}{q_2}\binom{2r - 2q_1 - 2q_2 - 2}{r - q_1 - q_2 - 1}$$

$$= \sum_{j=0}^{k}(r - 2j)^2, \qquad \left[\begin{array}{l} k = r/2 \text{ if } r \text{ is even and} \\ k = (r - 1)/2 \text{ if } r \text{ is odd} \end{array}\right]$$

$$= \frac{1}{6}r(r + 1)(r + 2)$$

as the number of common $(r-1)$-secant $(r-2)$-spaces of two normal curves of order r in r-space.* Thus, two twisted cubic curves in 3-space have 10 common secant lines.

As another application of the general formula (4) we give the following. Let

$$n_1+n_2+ \cdots +n_t=2r-4.$$

Hence, from (A), $s_t=2$. Then formula (4) gives the number of $(r-2)$-spaces that are incident with two given lines and meet t curves C^{m_i} n_i times where $\sum_{i=1}^{t} n_i=2r-4$. This is also the order of the hypersurface V_{r-1} formed by the ∞^1 $(r-2)$-spaces that are incident with a given line and meet C^{m_i} n_i times. The ∞^2 $(r-2)$-spaces incident with C^{m_i} n_i times meet a general 3-space, and in particular, a 3-space passing through l, in the lines of a congruence K the sum of whose order μ and class ν is the order of the hypersurface V_{r-1}. The order of K is evidently $N_{r-1}^{(t)}$, obtained from (4) by changing r to $r-1$, for this is the number of $(r-2)$-spaces that pass through a given point and meet C^{m_i} n_i times. Therefore, the class of K or the number of $(r-2)$-spaces that meet C^{m_i} n_i times and meet a given plane in lines is

$$\nu = N_r^{(t)} - N_{r-1}^{(t)}.$$

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* This result can also be obtained from (5) by putting $m_1=2r$ and $p_1=-1$.