ON THE DENSITY OF THE CUT POINTS AND END POINTS OF A CONTINUUM*

BY W. L. AYRES

1. *Introduction.* We consider a connected, compact, metric space M which we refer to as the continuum M. The space M is said to be *locally connected* if for each point p of M and each $\epsilon>0$ there exists a $\delta>0$ and a connected set N such that $S(p, \delta) \subset N \subset S(p, \epsilon)$. A point p is said to be a *cut point* of M if $M-p$ is not connected. A point p is said to be an *end point* of M if for each $\epsilon>0$ there exists a neighborhood U_p such that $U_p \subset S(p, \epsilon)$ and $B(U_p)=\overline{U}_p - U_p$ is a single point.† From this definition it is seen that every end point is a limit point of the cut points of M. Hence whenever the end points are dense in M, the cut points are also dense. This relation is not true conversely, but a study of some examples leads one to the conclusion that there exist some fundamental relations between the density of the cut points and the density of the end points. In this note we shall investigate some of these relations.

2. *Notation.* Let K and E denote respectively the set of all cut points and end points of M. Let K^2 denote the set of all cut points which are of Urysohn-Menger order 2 in M and let ${}^2K$ denote the set of all cut points which are of order >2. Capitals will denote sets of points, lower case letters single points. $S(p, \epsilon)$ denotes the set of all points whose distance from p is less than ϵ. The symbol $\rho(x, y)$ denotes the distance from x to y; $\rho(X, Y)$ denotes the greatest lower bound of the numbers $\rho(x, y)$ where $x \varepsilon X$ and $x \varepsilon Y$. The notations $x \varepsilon X$ and x non-$\varepsilon\, X$ mean "x is a point of the set X" and "x is not a point of X" respectively. The symbol $d(X)$ denotes the diameter of X, that is, the least upper bound of all numbers $\rho(x, y)$ where $x \varepsilon X$ and $y \varepsilon X$.

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† We use end point here in the Urysohn-Menger sense. See P. Urysohn, Comptes Rendus, vol. 175 (1922), pp. 481–483; and K. Menger, Mathematische Annalen, vol. 95 (1925), pp. 277–306. For other senses in which the term has been used, see H. M. Gehman, *Concerning end points of continuous curves and other continua*, Transactions of this Society, vol. 30 (1928), pp. 63–84.

3. THEOREM. *If the end points are dense in $M-K$, the cut points are dense in M.*

Since each end point is a limit point of the cut points, we have $\overline{E} \subset \overline{K}$. By hypothesis, $M-K \subset \overline{E}$. Then $M-K \subset \overline{K}$. As $K \subset \overline{K}$, we have $M=\overline{K}$.

4. THEOREM. *In order that the cut points of a locally connected continuum M be dense in M it is necessary and sufficient that the end points be dense in $M-K$.* $[(\overline{K}=M)\equiv(\overline{E}\supset M-K).]$

The condition is sufficient by the preceding theorem. Now let $p\varepsilon M-K-E$ and choose $\epsilon>0$. There exists just one maximal cyclic set C of M containing p.* By hypothesis there exists a $q_1'\varepsilon K\cdot S(p, \epsilon/2)$. If $q_1'\varepsilon C$, let $q_1=q_1'$ and let H_1 be a component of $M-q_1$ such that $H_1\cdot C=0$. If q_1' non-ε C, let H_1 be the component of $M-C$ containing q_1' and let $q_1 \varepsilon C\cdot\overline{H}_1$. Evidently $q_1=C\cdot\overline{H}_1$ and $q_1\neq p$. There exists an integer $m_2>2$ such that $S(p, \epsilon/m_2)\cdot H_1=0$. Let $q_2'\varepsilon K\cdot S(p, \epsilon/m_2)$. If $q_2'\varepsilon C$, let $q_2=q_2'$ and H_2 be a component of $M-q_2$ such that $H_2\cdot C=0$. If q_2' non-ε C, let H_2 be the component of $M-C$ containing q_2' and let $q_2=C\cdot\overline{H}_2$. We see quite easily that $H_2\cdot H_1=0$ and $q_2\neq p$. In general there exists an integer $m_i>m_{i-1}(i>1)$ such that $S(p, \epsilon/m_i)\cdot H_{i-1}=0$. Let $q_i'\varepsilon K\cdot S(p, \epsilon/m_i)$ and we find H_i and q_i from this exactly as above. Again we have $q_i\neq p$ and $H_i\cdot H_j=0$. Then we have an infinite set of distinct components of $M-C$, $H_1, H_2, H_3, \cdots$, such that

$$\overline{H}_i \supset q_i', \qquad \lim_{i\to\infty} q_i' = p, \tag{1}$$

$$\overline{H}_i\cdot C = q_i \neq p. \tag{2}$$

Since M is locally connected it follows from (1) and (2) that

$$\lim_{i\to\infty} q_i = p. \tag{3}$$

Now as $d(H_i)$ approaches zero it follows from (3) that there

* We are using the term *maximal cyclic set* in place of Whyburn's term *maximal cyclic curve.* For definitions and properties of maximal cyclic sets, see G. T. Whyburn, *Cyclicly connected continuous curves,* Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38. These results are extended to metric space in my paper, *Concerning continuous curves in metric space,* American Journal of Mathematics, vol. 51 (1929), pp. 577–594.

exists an integer n such that $S(p, \epsilon) \supset H_n$. If $\overline{H}_n$ is cyclicly connected, then $\overline{H}_n \cdot K = q_n$. Then no point of H_n is a limit point of K contrary to hypothesis. Then $\overline{H}_n$ is not cyclicly connected and thus contains two nodes of itself.* One of these nodes N does not contain q_n. Either N is an end point of $\overline{H}_n$ or N is a maximal cyclic set of $\overline{H}_n$ containing just one cut point of $\overline{H}_n$. But in this last case N is a maximal cyclic set of M and contains just one point of K. But this is impossible just as above. Hence N is an end point of $\overline{H}_n$ and it is not difficult to see that $N \varepsilon E$. Then for each $\epsilon > 0$ there exists an $N \varepsilon E \cdot S(p, \epsilon)$. Hence $p \varepsilon \overline{E}$.

5. *Example.* We may see quite simply that the condition of the preceding theorem is not necessary without the restriction that M be locally connected. Let M be the curve $y = \sin 1/x$ for $0 < x \leqq 1$ together with the points of the y-axis for which $-1 \leqq y \leqq 1$. Then $\overline{K} = M$ but the points of M on the y-axis belong to $M - K$ and no one of them belongs to $\overline{E}$. In fact E is the single point $(1, \sin 1)$.

6. THEOREM. *In order that the end points of a locally connected continuum M be dense in M it is necessary and sufficient that the cut points of order >2 be dense in M.* $[(\overline{E} = M) \equiv ({}^2\overline{K} = M).]$

The condition is necessary. Since $E \neq 0$, the continuum M is not cyclicly connected. Then if M contains a maximal cyclic set C, each component of $M - C$ has just one limit point in C and thus $C \cdot K \neq 0$. Further $C \cdot K = C \cdot {}^2K$, for if $x \varepsilon C \cdot K$ there exist at least three arcs meeting at x (two arcs on a simple closed curve of C containing x and one arc in a component of $M - C$ whose limit point is x) and so x is of order $\geqq 3$ in M.

Now let $p \varepsilon E$. If p non-ε ${}^2\overline{K}$, there exists a neighborhood U_p such that $U_p \cdot {}^2K = 0$. As $p \varepsilon E$ there exists a neighborhood V_p such that $\overline{V}_p \subset U_p$ and $B(V_p) = q$, a single point. Let A_1 be an arc of M with end points p and q and let $y \varepsilon A_1 - p - q$. Evidently $A_1 \subset \overline{V}_p$. From the local connectivity there exists a neighborhood $W_y \subset V_p - p$ such that if $z \varepsilon W_y$ then there is an arc from z to y which lies in V_p. As $\overline{E} = M$ let $z \varepsilon E \cdot W_y$. Then $z \varepsilon M - A_1$. Let A_2 be the arc of M joining z and y such

* G. T. Whyburn, *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), pp. 167–194, Theorem 14.

that $V_p \supset A_2$. In the order from z to y let w be the first point of A_1 on A_2. The point $w \neq p$ for $p \varepsilon E$ and thus is not interior to an arc of M. The point w is of order $\geqq 3$ since there are at least three arcs (subarc wp of A_1, subarc wq of A_1, subarc wz of A_2) meeting in w. Then w non-ε K since $V_p \cdot {}^2K = 0$. As w non-ε K there exists an arc $A_3 \subset M - w$ with end points z and p. It is easy to see that $\overline{V}_p \supset A_3$. From the set $A_1 + A_2 + A_3$ we may obtain a simple closed curve $J \subset \overline{V}_p$. Let C be the maximal cyclic set containing J. In the preceding paragraph we saw that $C \cdot {}^2K \neq 0$. But $C \subset \overline{V}_p \subset U_p$ and $U_p \cdot {}^2K = 0$. Then the assumption that p non-ε ${}^2\overline{K}$ leads to an absurdity. Hence $E \subset {}^2\overline{K}$. But $M = \overline{E}$. Then $M = {}^2\overline{K}$, which is the desired relation.

The condition is sufficient. From the result of §4 we have the set $F = M - K \subset \overline{E}$ and thus $\overline{F} \subset \overline{E}$. If C is a maximal cyclic set of M, all but a countable number of points of C belong to $M - K$. Then $C \subset \overline{F}$. Let $p \varepsilon M - \overline{F}$ which is of order 2 in M and let U_p be a neighborhood of p. There exists a number $\epsilon_1 > 0$ such that $S(p, \epsilon_1) \cdot \overline{F} = 0$ and $S(p, \epsilon_1) \subset U_p$. If J is any simple closed curve of M, $S(p, \epsilon_1) \cdot J = 0$. As $p \varepsilon K$ there is an arc $A \subset U_p$ and containing p as an interior point. Since ${}^2\overline{K} = M$, there is a point $q_1 \varepsilon {}^2K \cdot S(p, \epsilon_1)$. If $q_1 \varepsilon A$, let $x_1 = q_1$. In this case there is a component H_1 of $M - A$ such that $\overline{H}_1 \cdot A = q_1 = x_1$. If q_1 non-ε A, let H_1 be the component of $M - A$ containing q_1. The set $\overline{H}_1 \cdot A$ is a single point x_1 for otherwise there exists a simple closed curve J such that $J \cdot S(p, \epsilon_1) \neq 0$. Since p is of order 2, $p \neq x_1$. There exists a number $0 < \epsilon_2 < \frac{1}{2}\epsilon_1$ such that $S(p, \epsilon_2) \cdot \overline{H}_1 = 0$. Let $q_2 \epsilon {}^2K \cdot S(p, \epsilon_2)$. If $q_2 \varepsilon A$, let $q_2 = x_2$ and there is a component H_2 of $M - A$ such that $\overline{H}_2 \cdot A = q_2 = x_2$. If q_2 non-ε A, let H_2 be the component of $M - A$ containing q_2 and let $x_2 \varepsilon \overline{H}_2 \cdot A$. As above $x_2 = \overline{H}_2 \cdot A$ since $J \cdot S(p, \epsilon_1) = 0$ for every simple closed curve J. Continue this process indefinitely. In general there exists a number $0 < \epsilon_n < \frac{1}{2}\epsilon_{n-1}$ such that $S(p, \epsilon_n) \cdot (\overline{H}_1 + \overline{H}_2 + \cdots + \overline{H}_{n-1}) = 0$. Let $q_n \varepsilon {}^2K \cdot S(p, \epsilon_n)$ and define x_n and H_n as above. Then $x_n \neq p$.

From the connectedness im kleinen of M we have

$$\lim_{n \to \infty} x_n = p, \qquad \lim_{n \to \infty} d(H_n) = 0. \tag{4}$$

From this it follows that there exists an integer m such that $H_m \subset S(p, \epsilon_1)$. The locally connected continuum $\overline{H}_m$ is acyclic since $J \cdot S(p, \epsilon_1)=0$ for each simple closed curve of M. Then every node of $\overline{H}_m$ is an end point of $\overline{H}_m$. This set contains at least two nodes and thus there is a point $z \neq x_m$ which is an end point of $\overline{H}_m$. It is clear that $z \varepsilon E$. Then $z \subset \overline{H}_m \subset S(p, \epsilon_1) \subset U_p$. Therefore $p \epsilon \overline{E}$.

We have $M-K+K^2 \subset \overline{E}$, and $M-(M-K+K^2)={}^2K$. By a theorem of G. T. Whyburn,* the set ${}^2K$ is countable and thus contains no open subset of M. For this reason ${}^2K \subset \overline{E}$.

7. *The Necessity of Local Connectivity.* We shall show now that the condition of the preceding theorem is neither necessary nor sufficient unless M is locally connected. That the condition is not sufficient follows from the first two properties of a continuum M described in the next paragraph.

There exists a continuum M with the following properties: (1) *M has no end point,* (2) *the cut points of order >2 are dense in M,* (3) *the set of cut points of order* 2 *is an uncountable zero-dimensional set which is dense in M.*

Let a, b, c be the points $(0, 0)$, $(1, 1)$, $(1, -1)$. Let $M_1=abc$, where by such a symbol we will mean the set of all points of the triangle plus all interior points. Let x_i be the number $1-1/2^i$. Let y_i be the ith number in the series $0, \frac{1}{2}, 0, -7/8, \cdots$, that is, $y_i=0$, x_{i-1} or $-x_{i-1}$ according as i is of the form $2n+1$, $4n+2$ or $4n$. Let p_i $(i=1, 2, \cdots)$ be the point (x_i, y_i). Let u_i and v_i $(i=0, 1, 2, \cdots)$ be the upper and lower end points of a vertical segment of length $1/2^i$ with p_{i+1} as mid-point. Let $p_0=a$. Let

$$M_2 = bc + \sum_{i=0}^{\infty} p_i u_i v_i.$$

Now let T_i be the collineation such that $T_i(a)=p_i$, $T_i(b)=u_i$, $T_i(c)=v_i$. Let

$$M_3 = \sum_{i=0}^{\infty} T_i(M_2).$$

In general let Δ be a maximal triangle of M_{n-1} and p_Δ, u_Δ and v_Δ be the left-, upper right-, and lower right-hand vertices of Δ. Let T_Δ be the collineation such that $T_\Delta(a)=p_\Delta$, $T_\Delta(b)=u_\Delta$, $T_\Delta(c)=v_\Delta$, and let

* G. T. Whyburn, *Concerning collections of cuttings of connected point sets*, this Bulletin, vol. 35 (1929), pp. 87–104, Theorem 13.

$$M_n = \sum T_\Delta(M_2)$$

for every $\Delta \subset M_{n-1}$. Now let

$$G = \prod_{n=1}^{\infty} \overline{M}_n.$$

The set G is a continuum such that every vertical line $x=k(0 \leqq k \leqq 1)$ intersects it in a single point or in an interval which is a side of one of the triangles Δ at some stage. The mid-point of each of these intervals (except bc) is a cut point of G of infinite order due to the oscillation of the set to the left of the point. It is easily seen that these cut points are dense in G. The point a is the only end point of G. Let H be the reflection of G in the y-axis, and let $M=G+H$. Then the continuum M has properties (1) and (2).

Except for a countable number of values of k, the line $x=k(-1 \leqq k \leqq 1)$ intersects M in a single point. Let q be any such point. The point $q=a$ is obviously a cut point of order 2 as it is of order 1 in both G and H. If $q \neq a$, then q is interior to some such point. Then $q=a$ is obviously a cut point of order 2 as it is of order 1 in both G and H. If $q \neq a$, then q is interior to some triangle Δ of M_n at each stage. If $\epsilon > 0$, there exists an integer n such that the triangle Δ of M_n containing q is of diameter $< \frac{1}{2}\epsilon$ and a non-$\varepsilon\,\Delta$. The triangle Δ contains just two limit points of that part of M outside Δ (its left-hand vertex and the mid-point of the vertical side). From this it is easy to see that we may obtain a quadrilateral Q of diameter $<\epsilon$ which has just these two points in common with Δ, encloses the rest of Δ, and has in its exterior $M-\Delta$. Then the interior of Q is a neighborhood of q of diameter $<\epsilon$ whose boundary has just two points in common with M. Hence q is of order 2. That dim $K^2=0$ follows at once for both points of $M\cdot B(Q)$ belong to ${}^2K$. This gives $\dim_q K^2=0$ for $q \neq a$. It is also easy to see that $\dim_a K^2=0$.

We shall proceed to show now that the condition of the preceding theorem is not necessary unless M is locally connected, that is,

There exists a continuum M, which contains no cut point of order >2, and yet the end points are dense in M.

Let $[r_n]$ denote the rational points in the interval (0, 1), except 0, arranged in a sequence. Let $f_n(x) = (1/n^2)[\sin 1/(x-r_n)]$ for $x \neq r_n$; $f_n(x)=0$ for $x=r_n$ $(0 \leqq x \leqq 1)$. Let

$$F(x) = \sum_{n=1}^{\infty} f_n(x).$$

Let H be the set composed of the graph of $F(x)$ and its limit points.* The following properties of H may be easily verified: (a) H is a continuum, (b) every line $x=k(0\leqq k \leqq 1)$ intersects H in either a point or an interval, (c) every such point is a cut point of H of order 2, (d) the set of all such intervals is a countable set, (e) H contains no cut point of order >2, (f) the point $(0, F(0))$ is the single end point of H.

Now we may transform the set H into a set M_1 such that (1) the point $(0, F(0))$ of H becomes the point $a=(0, 0)$, (2) the upper and lower end points of the interval of H containing $(1, F(1))$ become the points $b=(1, 1)$ and $c=(1, -1)$, (3) every point of M_1 lies in or on the triangle abc, (4) the properties (a)–(f) remain true for M_1. In fact we may make this transformation so as not to change the x-coordinate of any point.

For each interval I_n of M_1 let W_n be a wedge-shaped region bounded by two planes through I_n making equal angles with the xy-plane, two planes through the end points of I_n perpendicular to I_n and a plane parallel to the xy-plane. Evidently the regions W_n may be taken so that

$$\overline{W}_i \cdot \overline{W}_j = 0 \ (i \neq j), \text{ and } d(W_n) < 1/n.$$

Let $(a_n, b_n, 0)$ and $(a_n, c_n, 0)$ be the end points of $I_n(b_n > c_n)$, and let d_n be a number such that the point $(a_n, \frac{1}{2}(b_n+c_n), d_n)$ lies in the interior of W_n. Let Ψ_n be the transformation $x'=a_n$, $y'=\frac{1}{2}(b_n+c_n)+\frac{1}{2}(b_n-c_n)y$, $z'=-d_nx+d_n$. Let

$$M_2 = \sum_{n=1}^{\infty} \Psi_n(M_1).$$

* See W. A. Wilson, *On the structure of a continuum, limited and irreducible between two points*, American Journal of Mathematics, vol. 48 (1926), p. 162. The continuum H may also be constructed by methods similar to the previous example by building up oscillations both right and left. However the description becomes rather intricate. Also the previous example may be set up by the Cantor method of condensation of singularities but its properties appear somewhat clearer by the use of the triangles Δ.

Let I_{nm} be the intervals of $\Psi_n(M_1)$ except for I_n, and for each I_{nm} we take a wedge-shaped region W_{nm} such that $\overline{W}_{ni}\cdot\overline{W}_{nj}=0(i\neq j)$, $d(W_{nm})<1/(n+m)$, $W_{nm}\cdot(M_1+M_2)=I_{nm}$, $\overline{W}_{nm}\subset$ interior W_n. Now we define a linear transformation Ψ_{nm} such that $\Psi_{nm}(M_1)$ is a set homeomorphic with M_1 and lying in W_{nm} except for I_{nm}, just as above. Let

$$M_3 = \sum_n \sum_m \Psi_{nm}(M_1).$$

Continue this process indefinitely. Now let

$$M = \sum_{i=1}^{\infty} M_i.$$

The set $\overline{M}$ is the required example. To simplify the description as far as possible we have defined $\overline{M}$ as lying in euclidean three-space. However, on careful examination of $\overline{M}$ it may be seen that $\overline{M}$ can be mapped in the plane.

8. THEOREM. *If the end points are dense in M, then* dim $M^2\leqq 0$.

In a paper not yet published,* I have proved that for the set M^2, that is, the points of M of order 2, we have the following relation:

$$M^2 = H + G,$$

where (1) dim $H\leqq 0$, (2) G is a vacuous or countable set of arcs A_i, (3) each arc-segment A_i is an open subset of M. Now if we suppose that dim $M^2>0$, it follows from this result that the set G contains at least one arc A_1. Let p be an interior point of A_1. From property (3) there exists a neighborhood $U_p\subset A_1\subset M^2$. Hence $U_p\cdot E=0$ contrary to the hypothesis of our theorem.

COROLLARY. *If the end points are dense in M, then* dim $K^2\leqq 0$.

9. THEOREM. *In order that the end points of a locally connected continuum M be dense in M it is necessary and sufficient that the cut points be dense in M and* dim $K^2\leqq 0$. $[(\overline{E}=M)\equiv(\overline{K}=M)\cdot(\dim K^2\leqq 0).]$

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* *On the regular points of a continuum.* This paper has been submitted to the Transactions of this Society. See an abstract in this Bulletin, vol. 36 (1930), p. 485.

The conditions of the theorem are necessary by the results of §3 and the corollary of §8. We may show that the conditions are sufficient by an argument following closely that used in proving the sufficiency of the condition in §6.

The first example of §7 shows that the conditions of the theorem are not sufficient if we do not specify that M is locally connected.

THE UNIVERSITY OF MICHIGAN

ON THE GENERALIZATION OF TRIGONOMETRIC IDENTITIES IN ARITHMETICAL PARAPHRASING*

BY H. T. ENGSTROM†

1. *Introduction.* Identities of the type

$$\sum_{s=1}^{m}\alpha_s \sin a_s x \equiv \sum_{t=1}^{n}\beta_t \sin b_t x, \tag{1}$$

where α_s, a_s, β_t, b_t are rational integers, arise in the comparison of like powers of the modulus when an elliptic function is represented in more than one way by trigonometric series. The following theorem is used in obtaining arithmetical results from such identities.

THEOREM 1. *If $g(x)$ is an arbitrary, single-valued, odd function, defined for $x=a_s$, $s=1, 2, \cdots, m$, and $x=b_t$, $t=1, 2, \cdots, n$, then* (1) *implies*

$$\sum_{s=1}^{m}\alpha_s g(a_s) = \sum_{t=1}^{n}\beta_t g(b_t). \tag{2}$$

Similarly, for cosines, we have the following statement.

THEOREM 2. *If $f(x)$ is an arbitrary, single-valued, even function, defined for $x=a_s$, $s=1, 2, \cdots, m$, and $x=b_t$, $t=1, 2, \cdots, n$, then*

$$\sum_{s=1}^{m}\alpha_s \cos a_s x \equiv \sum_{t=1}^{n}\beta_t \cos b_t x \tag{3}$$

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