equation of a matrix has the same roots as its minimum equation and, when the latter is irreducible, the former is an exact power of the latter. But for all values of the ξ_i in F the quantities $x=\sum\xi_i u_i$ are in a division algebra and have irreducible minimum equation. Hence $R(\omega;\ \xi_1, \cdots, \xi_m)=0$ is either irreducible in F when the ξ_i take on values in F or is a power of an irreducible equation and is irreducible when it has no multiple roots. But the discriminant $D(\xi_1, \cdots, \xi_m)$ of $R(\omega;\ \xi_i)$ is not identically zero, since $R(\omega;\ \xi_i)$ is irreducible in $F(\xi_1, \cdots, \xi_m)$. Hence* there exists an infinity of values of the ξ_i in F for which $D\neq 0$ and $R=0$, of degree n, is the minimum equation of the corresponding quantities x.

The proof of Hilbert's theorem is non-algebraic and even for fields of algebraic numbers it would be desirable to have an algebraic proof of our important theorem on normal division algebras. The above furnishes such a proof.†

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A COMMUTATION RULE IN QUANTUM MECHANICS

BY EUGENE FEENBERG

In a recent paper N. H. McCoy‡ has developed general commutation rules for the algebra of the quantum mechanics of Born, Heisenberg and Jordan. It is the purpose of this note to point out a commutation rule which in part is implicit in McCoy's work.

The fundamental equation of quantum mechanics from which the algebra is developed is

* See Fricke, *Algebra*, vol. I, p. 96, for a rational proof of this result which holds for any non-modular field F.

† The author wishes to take this opportunity to announce a correction of the results of his two papers in this Bulletin, vol. 35 (1929), pp. 335–338, and in the Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 372–376, respectively. In both of these papers the Hilbert theorem was used and the results of these papers are correct only for fields for which a *Hilbert irreducibility theorem* is provable. In the statement of Hilbert's theorem in the paper in this Bulletin, the reading should be "K any algebraic field over R, the field of all rational numbers," instead of "K any infinite field."

‡ *Algebra of quantum mechanics*, Transactions of this Society, vol. 31 (1929), pp. 793–806.

$$(1) \qquad pq - qp = cI,$$

in which the elements p, q are interpreted as matrices or as non-commutative operators which multiply associatively. I is the identity and c a number, real or complex, but not zero. Unique inverse elements are assumed to exist and are written q^{-1}, p^{-1}. This assumption is not fulfilled for all elements p, q satisfying equation (1), but by restricting exponents to positive values the results obtained are valid independent of the interpretation placed on p, q.

The result to be proved may be stated as follows:

$$(2) \quad (p^iq^i)^m(p^jq^j)^n(q^kp^k)^r(q^lp^l)^s = (q^lp^l)^s(q^kp^k)^r(p^jq^j)^n(p^iq^i)^m,$$
$$i,j,k,l,m,n,r,s = 0, \pm 1, \pm 2, \cdots .$$

By specializing the exponents it is seen that any pair of the terms $(p^iq^i)^m$, $(p^jq^j)^n$, $(q^kp^k)^r$, $(q^lp^l)^s$ commute.

Proof. Since $pq = qp + cI$, we have

$$(3) \quad (pq)(qp) = (qp)(pq), \quad (p^{-1}q^{-1})(q^{-1}p^{-1}) = (q^{-1}p^{-1})(p^{-1}q^{-1}).$$

Let us set $f = pqp^{-1}q^{-1}$. Then

$$fq^{-1}p^{-1} = (pq)(p^{-1}q^{-1})(q^{-1}p^{-1}) = (pq)(q^{-1}p^{-1})(p^{-1}q^{-1}) = p^{-1}q^{-1},$$

whence

$$f = (p^{-1}q^{-1})\ (q^{-1}p^{-1})^{-1} = (p^{-1}q^{-1})\ (pq).$$

It follows that

$$(4) \quad (pq)(p^{-1}q^{-1}) = (p^{-1}q^{-1})(pq), \quad (qp)(q^{-1}p^{-1}) = (q^{-1}p^{-1})(qp).$$

If G and H are functions of p, q commuting with pq, qp, $q^{-1}p^{-1}$, $p^{-1}q^{-1}$, GH commutes with pq, qp, $q^{-1}p^{-1}$, $p^{-1}q^{-1}$. Since $(pq)^{-n} = (q^{-1}p^{-1})^n$ and $(p^{-1}q^{-1})^{-n} = (qp)^n$ we find from (3) and (4) that

$$(5) \quad (pq)^m(qp)^n = (qp)^n(pq)^m, \qquad (n, m = 0, \pm 1, \pm 2, \cdots).$$

To complete the proof we show that

$$(6) \qquad p^nq^n = \sum_{i=0}^{n} a_{in}(pq)^i, \quad q^np^n = \sum_{i=0}^{n} b_{in}(qp)^i,$$

where $n = 0, 1, 2, \cdots$, and the coefficients a_{in}, b_{in} are numbers.

$$p^2q^2 = p(pq)q = p(qp + cI)q = (pq + cI)pq.$$

It follows by induction that

$$p^n q^n = (pq + (n-1)cI)p^{n-1}q^{n-1}, \qquad q^n p^n = (qp - (n-1)cI)q^{n-1}p^{n-1},$$

$n=0, 1, 2, \cdots$, from which (6) is readily proved. Then, by (5), we have

$$\begin{aligned}(p^n q^n)(p^m q^m) &= (p^m q^m)(p^n q^n),\\ (p^n q^n)(q^m p^m) &= (q^m p^m)(p^n q^n),\\ (q^n p^n)(q^m p^m) &= (q^m p^m)(q^n p^n), \qquad (m, n = 0, 1, 2, \cdots).\end{aligned} \tag{7}$$

From (7) we obtain

$$\begin{aligned}(q^{-m}p^{-m})(q^{-n}p^{-n}) &= (q^{-n}p^{-n})(q^{-m}p^{-m}),\\ (q^{-m}p^{-m})(p^n q^n) &= (p^n q^n)(q^{-m}p^{-m}),\end{aligned}$$

and four similar equations. Therefore (7) is true for $m, n=0, \pm 1, \pm 2, \cdots$, and since $(p^i q^i)^{-n} = (q^{-i}p^{-i})^n$ we have the result which was to be proved:

$$\begin{aligned}(p^i q^i)^m (p^j q^j)^n &= (p^j q^j)^n (p^i q^i)^m,\\ (p^i q^i)^m (q^j p^j)^n &= (q^j p^j)^n (p^i q^i)^m,\\ (q^i p^i)^m (q^j p^j)^n &= (q^j p^j)^n (q^i p^i)^m, \qquad (i, j, m, n = 0, \pm 1, \pm 2, \cdots).\end{aligned}$$

By specializing a general identity McCoy finds

$$q^m p^{m'} q^n p^{n'} = q^{n'} p^n q^{m'} p^m, \qquad m+n = m'+n'.$$

To illustrate the use of the theorem we verify this identity:

$$\begin{aligned}q^m p^{m'} q^n p^{n'} &= (q^m p^m)(p^{m'-m} q^{m'-m})(q^{n'} p^{n'})\\ &= (q^{n'} p^{n'})(p^{m'-m} q^{m'-m})(q^m p^m) = q^{n'} p^n q^{m'} p^m.\end{aligned}$$

If inverses do not exist and $m' < m$

$$\begin{aligned}q^m p^{m'} q^n p^{n'} &= q^{m-m'}(q^{m'} p^{m'})(q^n p^n) p^{n'-n}\\ &= q^{m-m'}(q^n p^n)(q^{m'} p^{m'}) p^{n'-n} = q^{n'} p^n q^{m'} p^m.\end{aligned}$$

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