ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume.* Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

317. Professor C. N. Moore: *On certain equivalent methods of summation.*

Given a series Σa_n and a positive monotonic function of n, λ_n, which becomes infinite with n. If $\lim_{n\to\infty}\Sigma_{m=1}^{n-1}(1-[\lambda_m/\lambda_n])^k a_m$ exists and is equal to A, we say that the series is summable (R, λ_n, k) to the value A. If $\lim_{\omega\to\infty}\Sigma_{m=1}^{\lambda_m<\omega}(1-[\lambda_m/\omega])^k a_m$ exists and is equal to A, we say that the series is summable (R, λ, k) to A. Both methods of summation are due to Marcel Riesz; the latter form is more generally used. It has been shown (Comptes Rendus, vol. 152 (1911), p. 1651) by Riesz that summability (R, λ, k), where $\lambda_n=n$, is equivalent to summability (C, k) for all real values of $k>-1$, and later (Proceedings of the London Mathematical Society, (2), vol. 22 (1923–24), p. 418) that summability (R, λ_n, k) for the same choice of λ is equivalent to summability (C, k) for $-1<k\leqq 1$. In the present paper it is shown that summability (R, λ_n, k) for the case $\lambda_n=c(n+q+[\psi_n/n])$, where ψ_n represents a function of n that remains bounded for all values of n and c and q are constants $(c>0)$, is equivalent to summability (R, λ_n, k) for $\lambda_n=n$, provided $-1<k\leqq 1$. In view of Riesz's second result, quoted above, this also involves the equivalence of the former type of summability with summability (C, k) if $-1<k\leqq 1$. (Received August 1, 1930.)

318. Professors Einar Hille and J. D. Tamarkin: *On the summability of Fourier series.* Third note.

The authors have continued their investigation of the effectiveness of the methods $[H, q(u)]$ of Hurwitz-Silverman-Hausdorff when applied to the summation of Fourier series. A distinction is made between F (Fejér) and L (Lebesgue) effective methods according as the applicability is ensured merely at regular points or almost everywhere. The authors exhibit sufficient conditions, in part also necessary, for both kinds of effectiveness. These conditions affect the integrability properties of the Fourier cosine transform of $q(u)$, and the continuity of $q(u)$ at the end points of the basic interval $(0,1)$. The paper also contains a study of the integrability properties of Fourier transforms. (Received August 7, 1930.)

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* See pp. 1–2 and p. 45 of the January issue.

319. Professor Dunham Jackson: *Note on the convergence of a sequence of approximating polynomials.*

Various results have been published with regard to the convergence of sequences of approximating polynomials defined by minimizing an integral which contains the mth power of the absolute value of the error, with a non-negative weight function. This note points out that the restrictions as to the vanishing of the weight function can be materially lightened, in comparison with the results cited, if it is assumed that the function to be approximated is analytic over a sufficiently extended region of the complex plane. (Received August 1, 1930.)

320. Professor R. L. Jeffery: *An application and an extension of Pierpont's theory of integration.*

In a short paper (Proceedings of the London Mathematical Society, (2), vol. 31, No. 1) we have discussed a problem stated in terms of Lebesgue's theory of integration for the solution of which this theory does not seem adequate. In the present paper we first show that the problem can easily be handled by means of Pierpont's theory. We then give a definition of an integral of a bounded function on a bounded set which is stated in terms of outer measure only, and show that this integral exists for an extensive class of non-measurable functions. (Received August 8, 1930.)

321. Professor F. L. Griffin: *As to a Goldschmidt solution for a wire of minimum attraction.*

In the problem of a wire exerting minimum attraction upon a particle at $O(0, 0)$, recently treated by Miss Swanson, no minimizing extremal can join $A(r_1, \theta_1)$ and $B(r_1, -\theta_1)$ in case $\theta_1 > 50°.24$, approximately. The present author points out that the analog of Goldschmidt's discontinuous solution for the minimal surface of revolution does not exist, though approximations thereto exist which give smaller attractions than certain combinations of extremal arcs in the case mentioned. (Received August 11, 1930.)

322. Professor T. R. Hollcroft: *Pencils of hypersurfaces.*

In a pencil of hypersurfaces in r-space, the number of hypersurfaces are found (1) that are tangent to a given manifold of any order and dimension; (2) that have a hypernode; (3) that have $r+1$ fixed points (or a hypernode at a fixed point) on a polar hypersurface; (4) whose Hessian passes through a fixed point; (5) that have contact with the Hessian. Involutions associated with the pencil and its polar hypersurfaces, the envelope of the Hessians of hypersurfaces of the pencil, the reduction in the number of hypernodes due to an s-fold basis point, are obtained. It is proved that the section of a general pencil of hypersurfaces in S_r by any linear space S_i, $i<r$, is a general pencil of hypersurfaces in S_i. (Received August 2, 1930.)

323. Dr. S. S. Cairns: *An axiomatic basis for euclidean plane geometry.*

Hilbert (*Grundlagen der Geometrie*, 1913, Anhang IV, pp. 163–218) has shown that three axioms pertaining to a set T of transformations carrying a plane into itself form a sufficient foundation for plane geometry. Various simplifications result from admitting orientation-reversing transformations and changing Hilbert's second and third axioms, as follows. Axiom 1: *The transformations T form a group.* Axiom 2: *Any two points, A and B, are invariant under some orientation-reversing transformation.* Axiom 3: *Any subgroup consisting of those transformations having two or more given fixed points in common is closed.* The points invariant under a transformation satisfying Axiom 2 constitute the *line AB*, and the transformation is called a *reflection.* As a consequence of the axioms, it is involutory. Hence two points uniquely determine a line. The images of a point B under all reflections leaving a point A fixed constitute a circle. Circles and lines are readily proved to be topologically equivalent to ordinary circles and lines in the (x, y)-plane. By some transformation, any line through a point A may be carried into any line through a point B in such a way that A goes into B. Other definitions and properties may readily be established. (Received August 11, 1930.)

324. Professor H. B. Curry: *Functionality in combinatory logic.*

The present paper is a development of combinatory logic (see abstracts 36-3-124 and 36-3-125) so as to apply to statements of the form "x belongs to the category α," where α is a proposition, function, Formel, Menge, or the like. An entity F (functionality) is introduced, such that the meaning of $(F\alpha\beta f)$, for instance, is "f is a function from α to β" or "fx belongs to β whenever x belongs to α," that of $(F\alpha(F\beta\gamma)f)$ is "fxy belongs to γ whenever x belongs to α and y to β," etc. What we may call the character of any entity is thus expressible by means of F and certain fundamental categories. Moreover an entity constructed by the processes already considered out of given entities whose characters are known will in general have a character dependent on the characters of the given entities. It is shown in this paper that any particular conclusion of this kind can be formally proved. The nature of the fundamental categories is irrelevant to this discussion and is not considered. However, a preliminary study of those paradoxes which are amenable to present treatment indicates that these all involve a fallacious assertion of the form under discussion. (Received August 7, 1930.)

325. Professor J. H. Neelley: *Concerning the possibility of a certain binary quartic being a line section of the plane quartic curve of genus zero.*

The nature of the roots of the catalectic cubic of the fundamental pencil have been established in this paper. The covariant quartics of the base quartics have been examined and one of them proves to be of special interest in that it can be a line section of the curve only if the curve osculates itself. Even in that case this covariant must be formed from a fundamental pencil with one catalectic base quartic. (Received August 11, 1930.)

326. Mr. A. L. Foster: *A continuous development of mathematical logic in finite terms.*

This paper develops the classical mathematical (formal) logic as a strictly finite discipline. Only finite sets are considered throughout, where intuitive clarity reigns. The usual axiomatic development, with all its multi-valued possibilities of expression (for example, *eq.* $\overline{p}v\overline{q}$, etc.) is discarded for a single-valued one, a refinement of the truth-table. All propositional functions (of the various types) are determined in terms of this representation, and all the truth-functions; the complete classification of the latter is reduced to a purely group-theoretical one. Among these possible truth-functions appear the classical $(Ex)F(x)$ and $(x)F(x)$ ("exists" and "every") but also others, for example the definite existence $(Dx)F(x)$, which is foreign to classical logic but is the existence (and the only one) recognized by the intuitionistic school. The theory of implication as that of pure deduction is developed, in which it appears that $(Ex)F(x)$ is empty in that nothing (non-trivial) can be deduced from it. Also $(Dx)F(x)\rightarrow(Ex)F(x)$, but not the reverse. The continuity of this development, as opposed to the classical one in which (Ex), (x) are inserted axiomatically, is basically responsible for the uncovering of fundamental facts. The whole language of formal logic is built up as a continuous, self-consistent, deductive system, without ever passing from the finite. (Received August 2, 1930.)

327. Professor W. A. Wilson: *A property of continua similar to local connectivity.*

The definition of local connectivity about a point in a continuous space Z is readily generalized to local connectivity about a sub-continuum x of Z. Recently G. T. Whyburn has extended the notion of ϵ-separability by a finite set of points to ϵ-separability by a finite set of continua. A bounded sub-continuum x of Z is called biregular if for every sub-continuum y of $Z-x$, Z is the union of two continua H and K such that $H\cdot x=K\cdot y=0$. Various properties of biregularity are proved and in particular it is shown that in strength biregularity lies between local connectivity and ϵ-separability. (Received August 4, 1930.)

328. Professor W. A. Wilson: *Semi-metric spaces.*

Let Z be a set of elements such that for each pair x and y there are two non-negative numbers xy and yx called the distances from x to y and from y to x, respectively, and satisfying the following axioms: (I) $xy=0$ if and only if $x=y$; (II) $xz\leqq xy+yz$. A space of this kind is called semi-metric. It is shown that these spaces have a large number of properties closely analogous to those of ordinary metric spaces and that they include two common types of topological spaces as special cases. (Received August 4, 1930.)

329. Professor G. T. Whyburn: *Concerning continuous curves without local separating points.*

In this paper it is shown that *if K is any closed totally disconnected subset of* *a continuous curve M having no local separating point and p and q are any two points of K, then there exists in M an arc pq containing K.* Let A denote the property of a continuous curve to contain, for each of its closed totally disconnected subsets K, an arc containing K; and let B denote the property of having no local separating point. According to the above theorem, property B implies property A. If each maximal cyclic curve of a continuous curve M has property B, then M has property A if and only if the cyclic elements of M form a simple cyclic chain. Clearly if a cyclically connected continuous curve is uni-coherent, it has property B. Thus the uni-coherent continuous curve M has property A if and only if the cyclic elements of M form a simple cyclic chain. Some necessary as well as the above sufficient conditions for property A are found. It is also demonstrated that every plane non-dense continuous curve having no local separting point is homeomorphic with the Sierpinski universal plane one-dimensional curve. (Received July 30, 1930.)

330. Professor G. T. Whyburn: *Concerning the subsets of regular curves.*

It is shown in this paper that every subset of a regular curve which is of dimension > 0 contains a non-degenerate connected set at points. As a consequence of this theorem it follows that all subsets of a regular curve (indeed, all regular point sets) may be divided into the following three mutually exclusive classes: (1) sets containing continua; (2) punctiform sets which contain non-degenerate connected sets; (3) zero-dimensional sets. It is also remarked that a more restrictive classification may be given for the regular sets K^n ($n = 1, 2, 3, \cdots$) consisting of all points of order exactly n of any compact and metric continuum. The set K^3 can be proved to be zero-dimensional. (Received July 30, 1930.)

331. Mr. C. M. Cleveland: *On the existence of acyclic curves satisfying certain conditions with respect to a given continuous curve.*

The author establishes the following theorem: If, in the plane, M is a bounded continuous curve which contains no domain and K is a totally disconnected closed subset of M, there exists an acyclic continuous curve T containing K and such that (1) all the end points of T belong to K, (2) the point set $M \cdot T$ is totally disconnected and $M-T$ is connected. (Received July 21, 1930.)

332. Dr. L. H. Bunyan: *Concerning a certain integral equation.*

The integral equation $u(x, y) = \lambda \iint_D K(x, y; \xi, \eta)\, u(\xi, \eta)\, d\xi d\eta$, when the kernel $K(x, y; \xi, \eta)$ has a logarithmic discontinuity at the point $(x, y) = (\xi, \eta)$, is transformed into a system composed of an integro-differential equation and a boundary condition. It is shown that this system is equivalent to the given integral equation in the sense that any solution of either is also a solution of the other. The integro-differential equation, without the boundary condition, is found to admit a solution analytic in λ and uniformly bounded for each value of the parameter λ which is sufficiently large in absolute value and located in a specified portion of the complex λ plane. (Received July 26, 1930.)

333. Dr. J. A. Shohat: *On certain differential equations.*

The classical orthogonal polynomials of Jacobi, Laguerre, and Hermite satisfy, it is known, a differential equation of the form (1) $Ay''+By'+Cy=0$, where A, B, and C are polynomials in x. The same holds for the orthogonal polynomials given by Heine (*Handbuch der Kugelfunktionen*, vol. 1, 1878), which it seems appropriate to call "elliptic polynomials," since they are closely related to $x=\sin \operatorname{am} \theta$, as the trigonometric polynomials are related to $x=\cos\theta$. In this paper the second particular solution of (1) is given in terms of the Tchebycheff function of the second kind, and also a differential equation is set up for the latter function. (Received August 5, 1930.)

334. Professor H. T. Davis: *The inversion of a differential operator of infinite order of Fuchsian type.*

This paper solves the following equation: (1) $A_0(x)u+A_1(x)xu'+A_2(x)\cdot x^2u''/2!+\cdots=f(x)$, where the $A_i(x)$ are analytic about $x=0$. The theories of the Volterra integral equation on the one hand and q-difference equations on the other are formally included in this generalization. The main results are as follows. A transformation exists and is explicitly given between the coefficients of $A_i(x)$ and the coefficients of a generalized Euler differential equation of infinite order such that if $v(t)$ is a solution of the latter, then $u(x)$, given by the inversion of the Laplace integral equation $v(t)=\int_L e^{xt}u(x)dx$, where L is a properly defined path, is a solution of the homogeneous case of (1). The solution of the hypergeometric equation about the origin and the solutions of the q-difference equation $u(qx)=(1-x)u(x)$ about $x=0$ and $x=\infty$ are found in this manner. The non-homogeneous equation is solved by a second transformation between the coefficients of (1) and a generalized Euler equation such that if $Y(x, z)$ is the resolvent generatrix of the latter then the resolvent of (1) is given by $X(x,z)=\Sigma_{n=0}^{\infty}Y^{(n)}(0, x)z^n/n!$. (Received August 4, 1930.)

335. Professor Marston Morse: *The problems of Lagrange and Mayer under general end conditions.*

A general formulation of problems of the above mentioned type has been given by Bolza, who obtained necessary conditions analogous to the Euler equations and the transversality conditions. Bliss lightened Bolza's hypotheses, and gave a new form to the problem. In its simpler aspects in the plane, it was recognized by Hilbert and others that the conditions analogous to the Jacobi conjugate point condition could be given in terms of characteristic roots of an auxiliary boundary value problem. A necessary condition of the latter sort for the general problem has recently been obtained by Cope. *Sufficient conditions in this general problem have never been established; the present paper establishes such conditions.* For an abstract of some of the results obtained, see Proceedings of the National Academy, vol. 16 (1930), p. 229, where references may also be found. (Received August 4, 1930.)

336. Professor Norbert Wiener: *Prime number theory and Planck's law.*

The problem of inverting Planck's law so as to obtain the distribution in temperature of a mixed black body radiation when its distribution in frequencies is known involves the study of the Riemann zeta function. In particular, the asymptotic study of this radiation for low frequencies, and consequently for low temperatures, involves identically the same mathematics as is needed to prove the prime number theorem. The Lambert series furnishes a link between the two. (Received July 24, 1930.)

337. Professor A. H. Copeland: *A matrix theory of measurement.*

The matrix theory of measurement furnishes a means of testing the consistency of the assumptions of the theory of measurement. It is a generalization of the theory of admissible numbers. In fact an admissible number may be considered a matrix each term of which is one or zero. We shall be concerned with infinite matrices whose terms are arranged in sequence. Such matrices will be called variates. If a variate satisfies certain assumptions of the theory of measurement, it is called admissible. With every admissible variate is associated an accumulative probability function. The mean value of a variate is defined as the limit, as n becomes infinite, of the average of its first n terms. An admissible variate possesses the property that the mean value of any continuous function of it is the Stieltjes integral of that function taken with respect to the accumulative probability function associated with it. Furthermore, the terms of an admissible variate must be independent. It is proved that corresponding to any denumerable set of accumulative probability functions there exists a set of independent admissible variates. These facts, although far from obvious, have been assumed in the theory of measurement. (Received August 9, 1930.)

338. Professor W. H. Durfee: *Summation factors which are powers of a complex variable.*

This paper deals chiefly with series of the form $y(z) = \Sigma a_n z^{f(n)}$, where Σa_n is summable with the value s by some Cesàro mean, z is a complex variable, and $f(n)$ belongs to the class of logarithmico-exponential functions. The theorems set forth conditions on $f(n)$ under which $y(z) \to s$ as $z \to +1$ from within the unit circle. These conditions differ according as the rate of increase of $f(n)$ is greater or less than that of $\log n$; in the latter case additional restrictions, including C_1-summability, are imposed on a_n. The latter part of the paper discusses certain series of the type $y(x) = \Sigma(-1)^n x^{f(n)}$ which, owing to too rapid a rate of increase of $f(n)$, do not approach a limit, but oscillate as $x \to +1$ from below. In this connection a generalization is given of a theorem on power series due to Belinfante (Koninklijke Akademie van Wetenschappen, Verslag, vol. 32(1923), p. 472). (Received August 11, 1930.)

339. Dr. F. W. Perkins: *Some properties of the oscillation of harmonic functions.*

In two earlier papers (*On the oscillation of harmonic functions*, Annals of Mathematics, (2), vol. 27(1925), pp. 159–170; *An intrinsic treatment of Poisson's integral*, American Journal of Mathematics, vol. 50(1928), pp. 389–414) the author has studied certain properties of the oscillation of harmonic functions, the discussion being directed primarily to the treatment of problems in the plane. The present paper gives the results of further study of this subject, with particular attention to problems in space. (Received August 11, 1930.)

340. Professor H. T. Davis: *The generalized Euler differential equation of infinite order.*

This paper gives the solution of both the homogeneous and non-homogeneous cases of the following differential equation of infinite order: $a_0u+(a_1+b_1x)u'+(a_2+b_2x+c_2x^2/2!)u''+\cdots=f(x)$, which includes the Euler differential equation as a special case. The essential contribution of this paper is the attainment of the general inverse operator for the non-homogeneous equation and the explicit solution of the homogeneous equation in the form $u(x)=\Sigma_{s=0}^{r}\, x^{\lambda+m_s}\log^{r-s}x(\phi_{s1}+\phi_{s2}/x+\phi_{s3}/x^2+\cdots)$, where λ is a root of a factorial series which generalizes the indicial equation of the differential equation of finite order with a regular singular point at the origin. When applied to the hypergeometric equation the theory of this paper gives the solution about the point $x=\infty$. In general no solution will exist about the origin, and an example is given of an equation with a solution about infinity for which the unit circle forms a natural bound. In illustration of the use of the inverse operator the Abel integral equation is solved by means of it. (Received August 4, 1930.)

341. Dr. Morris Marden: *On Stieltjes' differential equation.*

The differential equation under consideration is (1): $A(z)\,(d^2w/dz^2)+B(z)\cdot(dw/dz)+C(z)w=0$, where $A(z)$, $B(z)$, and $C(z)$ are polynomials of degrees p, $p-1$, $p-2$, and where $B(z)/A(z)=\Sigma_{j=1}^{p}\alpha_j/(z-a_j)$. Pólya has proved that, when the α_j are real positive numbers, the zeros of every polynomial solution of (1) lie in any convex region enclosing the points a_j. In its first part the present note shows that in this same convex region lie also the zeros of every polynomial $C(z)$ for which equation (1) has a polynomial solution. In its second part the present note generalizes Pólya's theorem to the case that the α_j are any complex numbers with positive real parts. (Received August 11, 1930.)

342. Professor C. O. Oakley: *Note on the methods of Sturm.*

The author considers the linear homogeneous differential equation of the second order in self-adjoint form. The point of view is that, simply by means of the Picone identity, oscillation and comparison theorems for the zeros of derivatives of solutions are directly developable. The theorems as stated apply both to zeros of derivatives of solutions and to zeros of solutions. (Received August 11, 1930.)

343. Professor J. L. Walsh: *Note on the overconvergence of sequences of polynomials of best approximation.*

This note is a continuation of some previous work by the present author. In notation previously used (Proceedings of the National Academy, vol. 16(1930), p. 297), the following result is proved. If the function $f(z)$ is analytic interior to C_R, then the sequence of polynomials of best approximation to $f(z)$ on C converges to $f(z)$ interior to C_R, uniformly on an arbitrary closed point set interior to C_R, (1) when C is an arbitrary limited rectifiable Jordan arc and best approximation is measured in the sense of least pth powers by a line integral along C, (2) when C is an arbitrary limited point set whose complement D (with respect to the entire plane) is simply connected and best approximation is measured in the sense of least pth powers by integration on the unit circle γ when D is mapped onto the exterior of γ so that the points at infinity correspond to each other. (Received July 17, 1930.)

344. Professor C. T. Holmes: *The approximation of harmonic functions in three dimensions by harmonic polynomials.*

The object of this paper is the investigation of conditions under which a harmonic function in three dimensions can be uniformly approximated by a harmonic polynomial, or by a harmonic rational function. It is shown that, if the boundary of a closed point set R satisfies a condition analogous to Raynor's criterion for the possibility of the solution of the Dirichlet problem, then any function continuous in R and harmonic in the interior can be approximated uniformly on R by a harmonic polynomial plus a finite number of harmonic functions each rational except for a factor which is the reciprocal of a distance. If the boundary of R is also the complete boundary of an infinite region, the approximating function is a harmonic polynomial. Attention is paid also to the manner in which the solution of the Dirichlet problem varies with the boundary of the region. (Received August 2, 1930.)

345. Dr. G. A. Hedlund: *Poincaré's rotation number and Morse's type number.*

The first points conjugate to the points of a periodic extremal in an ordinary problem in the calculus of variations define a transformation which is continuous and advances points so that order is preserved. Thus there is a rotation number corresponding to a periodic extremal. The type number of a periodic extremal is directly related to the number of conjugate points and a relation between these numbers is to be expected. This is established in this paper by proof that the rotation number determines the type number of not only one period, but of any number of successive periods. Conversely, the rotation number is determined by the sequence of type numbers of an increasing number of successive periods. The first part of the paper treats the orientable case, where the type number has been defined and its properties studied by Morse. In the second part, where the case of a non-orientable surface is considered, it is necessary to develop the theory of the type number. (Received August 5, 1930.)

346. Dr. G. A. Hedlund: *Geodesics on a two-dimensional Riemannian manifold with periodic coefficients.*

A geodesic segment on a manifold is of Class A if it affords an absolute minimum to all ordinary curves joining its end points and on the manifold. An unending geodesic is of Class A if any segment of it is of Class A. Given the positive definite differential form $ds^2=Edu^2+2Fdudv+Gdv^2$, where E, F, and G are periodic functions of class C^{iv} in u and v, it is proved that there exists a constant D, depending only on E, F, and G, such that, in the (u, v) plane, no Class A geodesic segment can wander a distance greater than D from the straight line joining its end points. With the aid of this theorem it is proved that there exists a Class A unending geodesic of the type of any given straight line in the (u, v) plane, and conversely, any one Class A unending geodesic is of the type of one straight line. A number of related theorems follow readily. The results apply to the geodesics on a surface homeomorphic with a torus. An example shows that the results of a similar nature do not hold in the case of more than two dimensions. (Received August 5, 1930.)

347. Dr. Lucien Féraud: *On Birkhoff's Pfaffian systems.*

This paper is a contribution to the study of the analytical and dynamical significance of the Pfaffian systems introduced by Birkhoff in his Colloquium Lectures. (Received July 26, 1930.)

348. Dr. S. B. Littauer (National Research Fellow): *On the symmetric form of the Fourier sine and cosine transforms.*

By the application of the Plancherel theory of Fourier transforms, the conventional definition of the sine and cosine transforms is so modified as to be distinguished only by a difference in a parameter. This form of definition lends itself to determining the asymptotic behavior of one of a pair of reciprocal functions so defined from the asymptotic behavior of the other. (Received August 8, 1930.)

349. Dr. R. P. Agnew (National Research Fellow): *The effects of general regular transformations on oscillations of sequences of functions.*

In a paper soon to appear in the Transactions of this Society, the author considers the behavior of continuous oscillation, continuous convergence, uniform oscillation, and uniform convergence of sequences of functions under regular transformations with triangular matrices. The present paper gives an extension of that investigation which employs general regular sequence-to-function transformations. Applications of regular transformations with square matrices, and of the Euler-Abel and Borel-Sannia transformations, are discussed. (Received July 22, 1930.)

350. Dr. N. H. McCoy (National Research Fellow): *Note on the existence of a positive function orthogonal to a given set of functions.*

A sufficient condition for the existence of a positive continuous function orthogonal to each member of a given set of functions $f_i(x)$, $i=1, 2, \cdots, m$, continuous and linearly independent on a closed interval X $(a\leqq x\leqq b)$, has been given by L. L. Dines (Annals of Mathematics, (2), vol. 28 (1926), pp. 393–395). The purpose of this note is to give two other sufficient conditions for the existence of such a function. Let $F_i^{(k)}(x) = \int_a^x F_i^{(k-1)}(x)\,dx$ $(i, k = 1, 2, \cdots, m)$, where $F_i^{(0)}(x) = f_i(x)$, and denote by $(-1)^k D_k(x)$ the determinant obtained from the matrix $(F_i^{(j)}(x))$ $(i=1, 2, \cdots, m; j=0, 1, \cdots, m)$ by striking out the kth column. It is shown that if there exists a value of x on X for which the $D_k(x)$ are all different from zero and of the same sign, then the given set of functions admits a function with the desired property. The other condition involves the existence of $m+1$ distinct values of x satisfying certain requirements. (Received August 11, 1930.)

351. Professor H. S. White: *The construction of a headless and groupless triad system on* 31 *elements.*

Triad systems on 15 elements may have no head, that is, contain no complete and closed system of triads on 7 elements, and may still admit a group of transformations into themselves; or they may be both groupless and headless. Of the latter sort are, for example, the last three systems discovered by F. N. Cole, Nos. 34, 35, 36. For 31 elements the existence of groupless triad systems was demonstrated and millions of them, all incongruent, were constructed, but all of them had heads, closed systems on 15 elements. The present note modifies one of these, namely one based on Cole's No. 36, and proves that the resulting Δ_{31} is both groupless and headless. (Received August 8, 1930.)

352. Professor A. J. Kempner: *Residue systems for a prime modulus.*

Let the element a_{ij} in a square array of p rows and columns be the residue, modulo p, of j^i. This table shows a greater degree of symmetry than seems to have been noticed, and permits the derivation of relations between the residues of a^i, a fixed, and j^c, c fixed. We deal with relations between certain subsets of these residues, rather than with the individual residues. In the second part of the paper contact is established with the "completely reduced polynomials" investigated by the author in previous papers. In particular, a theory of factorization for such polynomials in one or more variables is established. The results are at present restricted to a prime modulus. (Received August 9, 1930.)

353. Dr. H. T. Engstrom (National Research Fellow): *On sequences defined by linear recurrence relations.*

Periodicity for a rational integral modulus m in a sequence of rational integers defined by a linear recurrence relation of kth order has been studied by Carmichael for moduli m whose prime divisors exceed k. In the present paper the author obtains a solution of the problem without restriction on m by use of the theory of ideals. The results obtained include those of Carmichael. (Received August 11, 1930.)

354. Professors C. L. E. Moore and Philip Franklin: *Algebraic Pfaffians.*

This paper is devoted to a study of the geometric interpretation of the vanishing of a special type of Pfaffian. The coefficients are polynomials in the variables, which are thought of as Cartesian coordinates in space of three dimensions. At times, however, homogeneous coordinates are used, and some of the results are extended to n variables. When the polynomials are at most of the second degree, there is a certain complex of straight lines included among the integral curves. These complexes are studied, and made the basis of a classification of the possible types of Pfaffians under discussion. (Received July 23, 1930.)

355. Professor R. D. Carmichael: *Expansions of arithmetical functions in infinite series.*

The remarkable expansions of arithmetical functions obtained by Ramanujan in his notable memoir (Collected Papers, No. 21) on certain trigonometrical sums and their applications are contained as special cases of much more general expansions which have also other special cases of particular interest. It is the purpose of this paper to present these generalizations and to draw from the expansions some conclusions of importance obtained by means of a hitherto unnoticed fundamental property of the Ramanujan sums $c_q(n)$, namely, that expressed by the relations $\sum_{n=1}^{q\rho} c_q(n)c_\rho(n)=0$ if $\rho\neq q$, $\sum_{n=1}^{q} c_q^2(n)=q\phi(q)$. This property has led to the notion of orthogonal arithmetical functions analogous to the notion of orthogonal functions in analysis. The most characteristic results of the paper are those relating to the asymptotic forms of certain functions defined by aid of the divisors of a given integer. The functions involved here have several applications which we hope to present later and in particular an important application to the problem of the representation of integers as sums of squares. (Received August 4, 1930.)

356. Professor R. D. Carmichael: *On the representation of integers as sums of an even number of squares or of triangular numbers.*

Let s and n be positive integers, $r_{2s}(n)$ the number of ways of representing n as a sum of $2s$ squares, and $\gamma_q(n)=\cos\left[\frac{1}{2}\pi s(q-1)\right]\sum_\lambda\cos(2\pi\lambda n/q)-\sin\left[\frac{1}{2}\pi s(q-1)\right]\sum_\lambda(-1)^{(\lambda-1)/2}\sin(2\pi\lambda n/q)$, $q=1, 2, 3, \cdots$, where the sums are taken for the positive integers λ which are prime to q and do not exceed q. This paper proves the formula $(1/m)\sum_{n=1}^{m}\gamma_\rho(n)n^{1-s}r_{2s}(n)=\pi^s\epsilon_\rho(s)\phi(q)/[(s-1)!\rho^s]+O(1/m)$, $s>3$, where ρ is any positive integer, $\phi(\rho)$ is the Euler ϕ-function and $\epsilon_\rho(s)$ is 1, 0, or 2^s according as ρ is odd, $\rho\equiv 2 \bmod 4$, or $\rho\equiv 0 \bmod 4$. For $\rho=1$ this says that the function $n^{1-s}r_{2s}(n)$ is in the mean (on the average) equal to $\pi^s/(s-1)!$ if $s>3$. Similar results are obtained for the function $\sum_{r,s}(n)$ of Ramanujan (Collected Papers, p. 136) and for certain functions connected with the problem of representing integers as sums of triangular numbers. This paper is based on important results due to Ramanujan (loc. cit., pp. 136–137, 179–199) and on a certain hitherto unnoticed property of orthogonality of the Ramanujan functions $\gamma_q(n)$. (Received August 4, 1930.)

357. Professor R. D. Carmichael: *Concerning quasi-k-fold transitivity of permutation groups.*

Let G be a permutation group having the property that for every l such that $1 \leqq l \leqq k$ it is true that when two sets are given of l letters each, taken from the letters on which G operates, then a permutation P exists in G which transforms one of these sets in some order into the other; then it will be said that G is quasi-k-fold transitive. It is clear that G is transitive in the ordinary sense. A general theorem concerning the order of such groups is proved. Infinite classes of singly transitive and doubly transitive groups are exhibited such that they are respectively quasi-2-fold and quasi-3-fold transitive. A single example is given of a three-fold transitive group which is quasi-4-fold transitive. These examples arise from the linear fractional group in the Galois field $GF[p^n]$. (Received August 4, 1930.)

358. Professor R. D. Carmichael: *On a question related to Waring's problem.*

Let D_m $(m > 1)$ denote the multiplicative domain which is generated by 0, 1, the prime factors of m, and the νth powers of the primes of the form $mx+\alpha$ where α ranges over the integers less than m and prime to m, and ν is the exponent to which α belongs modulo m. For such a domain the following questions (suggested by Waring's problem) are proposed: What is the least integer $g(m)$ such that every positive integer is a sum of $g(m)$ integers belonging to D_m? What is the least integer $G(m)$ such that all but a finite number of positive integers are each expressible as a sum of $G(m)$ integers belonging to D_m? What is the least integer $\Gamma(m)$ such that "nearly all" positive integers are each expressible as a sum of $\Gamma(m)$ integers belonging to D_m? These questions seem difficult in the general case. The following theorems are proved: (I) $g(3) = G(3) = \Gamma(3) = 2$; (II) $g(4) = G(4) = \Gamma(4) = 2$; (III) $g(6) = G(6) = \Gamma(6) = 2$; (IV) $g(8) = G(8) = \Gamma(8) = 3$; (V) $g(12) = 3$, $G(12) = 2$ or 3, $\Gamma(12) = 2$ or 3; (VI) $g(24) = G(24) = \Gamma(24)$. Some empirical results are indicated for other cases. Certain algebraic forms associated with particular domains D_m are also investigated. (Received August 4, 1930.)

359. Professor E. T. Browne: *The characteristic roots of a matrix.*

This paper is concerned with the problem of finding an upper limit to the roots of the characteristic equation of a matrix A, real or complex, of order n, a problem which has engaged the attention of Bendixson, Hirsch, Bromwich, and others. By a remarkably simple method the author establishes the following theorem: If $S_i(T_i)$ is the sum of the absolute values of the elements in the ith row (column) of a square matrix A, and if $S(T)$ is the greatest of the $S_i(T_i)$, the absolute value $|\lambda|$ of a characteristic root λ of A cannot exceed $(S+T)/2$. Then by invoking results due to Hirsch and Bromwich, upper limits to the real and imaginary parts of λ are found. (Received July 15, 1930.)

360. Professor E. T. Browne: *On the separation property of the roots of the secular equation.*

Let A be an Hermitian matrix of order n and I the unit matrix. If $L_i(\lambda)$ is the principal minor determinant of order i standing in the upper left hand corner of $A - \lambda I$, it is well known that the roots of $L_i(\lambda) = 0$ are all real and in general separate the roots of $L_{i+1}(\lambda)=0$. However, in case the former equation has a multiple root the sense in which this separation takes place is not exactly clear, since a root of multiplicity m of $L_i(\lambda)=0$ may be a root of multiplicity $m-1$, m, or even $m+1$ of $L_{i+1}(\lambda)=0$. In this paper a study is made of the separation property in the case where $L_i(\lambda)=0$ has multiple roots. (Received July 15, 1930.)

361. Professor C. O. Oakley: *Semi-linear integral equations.*

We shall write as the type-form of the equations treated in this paper $y(x)+\sigma g(x)\left|y(x)\right|=f(x)+\lambda\int_\alpha^\beta K(x,\xi)y(x)d\xi+\mu\int_\gamma^\delta L(x,\xi)\left|y(\xi)\right|d\xi$. Existence theorems are derived by the method of successive approximations for small values of the parameters σ, λ, μ and with suitable restrictions upon g, f, K, L. The analogy with the linear case which is included is apparent. Outside of small circles, however, this analogy breaks down, for it is found that under certain conditions there is no solution for $\left|\mu\right|>c$, where c is a known constant. Further theorems of restriction show that in certain cases the region of existence of solutions may even be non-circular. (Received August 11, 1930.)

362. Dr. J. J. Gergen (National Research Fellow): *Convergence and summability criteria for Fourier series.*

This paper gives first a generalization (Theorem I below) of the classical Lebesgue criterion for the convergence of a Fourier series. It then discusses the relation of this test to the six commonly recognized criteria and certain of their generalizations. It is found that in this form the Lebesgue test strictly includes these others, in particular, Hardy's and Littlewood's. Theorem I is then extended in two directions. A slightly more general convergence criterion is obtained, and also a criterion for Cesàro summability of arbitrary order. It is proved that in one of the alternative forms of Lebesgue's original theorem the continuity condition is redundant. Theorem I may be stated as follows. Suppose that $f(t)$ is integrable (Lebesgue) over $(0, 2\pi)$, and periodic with period 2π. Then, if at the point $t=t_0$ the Fourier series of $f(t)$ is summable, to sum s in some Cesàro manner, and if (1) $\lim_{k\to\infty}\limsup_{x\to+0}\int_{kx}^{\pi}\left\{\left|\phi(t+x)-\phi(t)\right|/t\right\}dt=0$, where $\phi(t)=\left\{f(t_0+t)+f(t_0-t)\right\}/2$, the series converges there, to sum s. Moreover, when (1) holds, it is necessary and sufficient for summability, to sum s, at $t=t_0$, that $\lim_{x\to+0}\left\{x^{-1}\int_0^x\phi(t)dt\right\}=s$. (Received August 11, 1930.)

363. Professor B. F. Kimball: *A useful formula in the theory of finite differences.*

A formula showing a relation between the rth difference $\Delta_r f(x)$ and the rth derivative $f^{(r)}(x+\frac{1}{2}r)$ of a function is demonstrated, and certain related theorems and applications are discussed. (Received August 11, 1930.)

364. Dr. Morris Marden: *A rule of signs involving certain orthogonal polynomials.*

In his Oeuvres, vol. 1, pp. 144–146, Laguerre gives a rule for determining the number N of real zeros greater than or equal to unity of a given real polynomial $F(x)$. Namely, express $F(x)$ in terms of the Legendre polynomials

$P_j(x)$: $F(x)=c_0P_0(x)+c_1P_1(x)+\cdots+c_nP_n(x)$, and calculate the number V of variations of sign in the sequence $c_0, c_1, \cdots, c_n$. Then $N\leqq V$. The present note gives a similar rule, wherein the $P_j(x)$ are a set of any Jacobi or generalized Laguerre polynomials. (Received August 11, 1930.)

365. Dr. Dorothy W. Weeks: *A study of the interference of polarized light by the method of coherency matrices.*

Wiener has developed the method of analyzing polarized light by coherency matrices, and applied it to the case of a single polarized beam of homogeneous light. This paper extends it to the case of two simultaneous beams of homogeneous light. The general form for this matrix is given, and sixteen fundamental coherency matrices are made the basis of study. The transformation matrices that change one coherency matrix into another are determined. Eight of the coherency matrices represent types of incoherent and eight of coherent light. The transformation matrices fall into two main classes, the β and ϕ classes. The β class has three main subclasses, each representing a closed group of thirty-two elements. In determining the β class, twenty-four possible forms in three sets of eight satisfy the imposed conditions. These twenty-four forms constitute a closed group, reducible to a four, a three, and a two group. The three group contains a form from each of the three sets of eight, and has therefore been selected. The forms of the four group are the forms of the Dirac matrices. Since the coherency matrices represent different types of light, the transformation matrices must characterize the optical instruments that change one type of light into another. Michelson's interferometer can be represented by such a matrix. (Received August 1, 1930.)

366. Professor R. E. Langer: *On the flow of heat for a solid in contact with a liquid.*

A cylindrical solid of length a in direction x is placed in contact at its face $x=a$ with a stirred liquid extending to $x=a+b$. The lateral surface is insulated. Heat may or may not be permitted to flow across the faces $x=0$, $x=a+b$. The initial temperatures in the solid and the initial temperature of the liquid being given, the subsequent temperatures are to be determined. The following points are of interest. (1) The method of Fourier leads to an expansion in non-orthogonal functions. This point has led previous investigators to an entire change of method (H. W. March and Warren Weaver, Physical Review, vol. 31 (1928), p. 1072) or to the use of special devices (R. L. Peek, Jr., Annals of Mathematics, vol. 30 (1929), p. 265). It is here shown to be amenable to systematic treatment. (2) If the initial temperature of the liquid differs from that of the solid the solution must be discontinuous at $t=0$. This point (ignored by some authors) explains physically a familiar pecularity of behavior of certain series expansions at the ends of the convergence interval. The problem can be phrased also as a problem in diffusion. (Received August 2, 1930.)

367. Professor K. W. Lamson: *Some differential and algebraic consequences of the Einstein field equations.*

If a set of four directions in a four-dimensional Riemannian space is orthogonal, then the ds^2 can be expressed in terms of their sixteen parameters. The first purpose of this paper is to set up sixteen invariant linear first-order partial differential equations in these parameters. The solutions of these equations include all non-degenerate solutions of the Einstein field equations of 1917. The components of the curvature tensor may be taken as coefficients of the equation of a quadratic line complex in a three-dimensional space. The second purpose of the paper is the application of some of the theory of complexes to the theory of spaces satisfying the Einstein equations. (Received August 5, 1930.)

368. Mr. Nicholas Rashevsky: *Some new mathematical problems in biophysics.*

The theoretical study of certain physical phenomena which are supposed to occur in living organisms leads to various new types of mathematical problems, not hitherto studied in pure mathematics. As examples, two such problems are discussed. One relates to the determination of the shape of equilibrium of liquid drops, in which certain capillary active substances are constantly diffusing, causing a non-uniformity in the distribution of the surface tension along the surface. This problem has been sketched in a previous publication (Zeitschrift für Physik, vol. 56 (1929), p. 297) and is somewhat analogous to that of determining the unknown kernel of an integral equation from the known solution. Another still more unusual type of problem arises in connection with the theory of propagation of the nervous impulse. A few other problems are briefly mentioned. (Received August 11, 1930.)

369. Mr. B. F. Groat: *Mean value of the ordinate of the locus of the rational integral algebraic function of degree n expressed as a weighted mean of $n+1$ ordinates, and the resulting rules of quadrature.*

There are several well known rules, such as Newton's, Simpson's (or Cotes'), and others, for approximating to the areas of closed regions; and such approximations have been studied by Newton, Lagrange, Gauss, and others. The author of this paper does not know of any general formulas for finding the averaging weights of arbitrarily chosen ordinates which will give the true area under the general curve of rational integral algebraic form, nor of any rule for finding the mean ordinate of such a curve as the arithmetic mean of a determinate number of properly situated ordinates. In the present paper, the author derives rules and formulas for these purposes. (Received August 2, 1930.)

370. Dr. W. D. Baten: *Corrections for the moments of a frequency distribution in two variables.*

In certain statistical problems it is beneficial to divide the given data into classes or groups and then investigate the distribution in this form. The moments determined for the distribution divided into classes differ from the moments determined from the original data. It is the object of the present paper to show how to modify the former to secure the latter. The corrections for the moments are obtained by use of the Euler-Maclaurin summation formula for two variables and certain assumptions concerning the frequency function at the limits of the distribution. (Received August 11, 1930.)

371. Professor H. W. Sibert: *Moderately thick circular plates with plane faces.*

The method of solution of the above mentioned problem used in this paper is based on the assumption that the displacements may be expanded in positive integral powers of z, the coefficients being polynomials in r. An infinite set of constants is developed which makes it possible to prove convergence for a limited class of loading functions. This is the first time that convergence has been proved for a method based upon expansion in positive integral powers of r and z. Nádai and Clemmow have each given solutions involving Bessel functions which are convergent for a limited class of loading functions, but the calculations involved are so difficult that it would be almost impossible to extend their methods to the more complicated types of loading conditions that can be treated by the method of this paper. A general formula for the displacements is obtained, from which a solution may be derived for any loading independent of θ which results in convergent series for the displacements. In applying this method to physical problems, it has been possible to obtain all the constants of integration by a rigorous mathematical treatment not dependent on physical intuition. (Received July 16, 1930.)

372. Miss Marian A. Wilder: *Correlation coefficients and transformation of axes.*

It is well known that coefficients of correlation can be interpreted as cosines of angles, either as a direct application of the fundamental formulas of analytic geometry in space of n dimensions, or, in the case of a normal frequency distribution, in connection with transformation of the axes of reference for the ellipsoids of equal frequency. This paper shows how a configuration corresponding to that which forms the basis of the latter interpretation, and reducing to it in the case of a normal distribution, can be defined for a frequency distribution of arbitrary form. (Received August 1, 1930.)

373. Dr. A. C. Berry (National Research Fellow): *Linear transformations in conjugate abstract spaces.*

M. Riesz (Acta Mathematica, vol. 49 (1926), pp. 465–497) has studied the convexity properties of the bounds of linear transformations operating on functions whose pth powers possess a Lebesgue-Stieltjes integral with respect to a given non-decreasing function. Conjugate exponents p, p' play an important rôle. In the present paper there are introduced general notions of convexity and of conjugate abstract spaces. The important theorems of Riesz are generalized and the proofs in some instances simplified. Following Riesz's suggestion the theory of the Fourier transform is given as an application and is thus established without reference to the theory of Fourier series. (Received August 16, 1930.)

374. Dr. Hillel Poritsky: *On conservative transformations connected with certain harmonic functions.*

By a conservative transformation of a surface into itself is meant (following Birkhoff) a transformation possessing a positive invariant surface integral. The importance of such transformations in dynamics has been shown by the work of Poincaré and Birkhoff. In this note attention is called to certain conservative transformations connected with harmonic functions of the form $v=\sum_{i=1}^{k}c_i/r_i$, where the c_i are constants and the r_i represent the distances from k fixed points, P_i, respectively. The transformation is that of a complete equipotential surface $v=v_0$ into itself; the transform $T(P)$ of a point P of $v=v_0$ is obtained by following along the line of force through P, first in the direction of, say, increasing v, through one of the points P_i or the point at infinity, and until the line of force intersects $v=v_0$ again; the latter point of intersection is $T(P)$. This transformation is one to one and continuous except possibly at the points which lie on lines of force passing through points of vanishing force, and admits as invariant integral the "flux" integral $\int(\partial u/\partial n)dS$. If the constants c_i are all positive and v_0 is small enough, $v=v_0$ is approximately spherical; when v_0 approaches 0, a transformation of a sphere into itself is obtained which preserves areas. (Received August 12, 1930.)

375. Professor W. H. McEwen: *Problems of closest approximation connected with the solution of linear differential equations.*

This paper is concerned with the approximate representation of a solution of a given linear ordinary differential equation of the mth order with boundary conditions, by means of linear combinations of given functions, particularly by means of trigonometric sums and polynomials, the approximating functions being subjected to the boundary conditions of the differential system. The criterion of approximation is the minimizing of the integral of $|L(y)-L(y_n)|^r$, where y is the function to be approximated, $L(y)$ the differential expression forming the left-hand member of the differential equation, and y_n the approximating function. Related problems have been discussed recently by Kryloff, Picone, and others, but the methods and results are different. The convergence proofs for $r>1$ are based on a simple and direct application of Hölder's inequality, and for $r<1$ on an extension of Bernstein's theorem on the derivative of a trigonometric sum or polynomial. (Received August 21, 1930.)

376. Professor O. D. Kellogg: *Capacity of sets of Cantor type.*

The fundamental lemma that every closed bounded set of positive capacity has a regular point has been proved for the logarithmic potential (Comptes Rendus, vol. 187 (1928), p. 526), but not as yet in space of three dimensions. The importance and difficulty of the problem make special cases of interest (see American Journal of Mathematics, vol. 51 (1929), pp. 515–526). Among sets for which the lemma might seem most likely to fail are those of Cantor type formed as follows. From the unit cube are removed all points whose distances from the planes through the center and parallel to the sides are less than $\alpha_1/2$, the remaining points constituting the set E_1. From the cubes of E_1 are removed all points whose distances from the corresponding planes are less than $\alpha_2/2$

times the length of the sides of the cubes, all the remaining cubes constituting the set E_2, and so on. The set E consisting of all the points common to the infinite sequence of sets $E_1, E_2, E_3, \cdots$ is the set of Cantor type corresponding to the sequence of positive proper fractions $\alpha_1, \alpha_2, \alpha_3, \cdots$. Such sets may have positive, or zero, capacity, according to the sequence of α's. But if this sequence is either bounded away from 0, or converges to 0, the fundamental lemma holds for the set. It is also shown that sets with positive outer Jordan content always contain regular points. (Received August 21, 1930.)

377. Professor H. S. Wall: *On the Padé approximants associated with certain divergent series.*

The following theorem applies to (a) Stieltjes series, (b) positive definite series, (c) certain normal series having corresponding continued fractions

$$\frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \cdots$$

in which $\sum |a_i|$ converges. Theorem: *Let the principal diagonal file, S_0, of the associated Padé table, and the first parallel file, S_{-1}, below S_0, converge and have different limits. Then an arbitrary file S_k converges to a limit of the form u_k/v_k, where $u_k = \alpha_k p - \beta_k p_1$, $v_k = \alpha_k q - \beta_k q_1$; p, q, p_1, q_1 are entire functions; α_k, β_k are polynomials in $1/z$; $\alpha_{-1}=1$, $\beta_{-1}=0$, $\alpha_0=0$, $\beta_0=-1$; $u_k v_{k'} - u_{k'} v_k = \alpha_k\beta_{k'} - \alpha_{k'}\beta_k$.* Let $\bar{u}_k = u_k/(\alpha_k - \beta_k)$, $\bar{v}_k = v_k/(\alpha_k - \beta_k)$. Then if z is real, the points $(\bar{v}_k, \bar{u}_k)$, $k=0, \pm 1, \pm 2, \cdots$, lie on a straight line, $L(z)$, which does not pass through the origin. For (a), (b), $L(x)$ rotates continuously clockwise about the origin as x increases from $-\infty$ to $+\infty$. (Received September 4, 1930.)

378. Dr. C. I. Lubin: *A further note on singular points of the differential equation $dx/X(x, y) = dy/Y(x, y)$.*

In the system $dx/dt = X(x, y)$, $dy/dt = Y(x, y)$, the real functions $X(x, y)$ and $Y(x, y)$ of the real variables x and y are supposed to be analytic in the neighborhood of the origin and to vanish at the origin. The matrix of the linear terms of the power series expansions of $X(x, y)$ and $Y(x, y)$ is assumed to be of rank 2 and to have characteristic roots $\pm\lambda(-1)^{1/2}$. This discussion is preliminary to further investigation regarding the nature of the functions accomplishing the transformation. The form attained by the formal reduction is $du/dt = v + \frac{1}{2}[Av - Bu](u^2+v^2)^m - \frac{1}{2}C(u^2+v^2)^{2m}u$, $dv/dt = -u - \frac{1}{2}[Au+Bv] \cdot (u^2+v^2)^m - \frac{1}{2}C(u^2+v^2)^{2m} v$, where if $B = C = 0$ the case of centers arises. This form can be further reduced to one with $A = 0$ by change of parameter. (Received September 5, 1930.)

379. Dr. W. J. Trjitzinsky (National Research Fellow): *A solution of a problem of Birkhoff.*

In 1913 Birkhoff stated, without proof, a theorem which involves the question of the possibility of constructing a function analytic in an open interval (0,1) and having the following properties: the function, together with its derivatives of all orders, has assigned values at the end points of this interval; moreover, it is a sufficient approximation to a prescribed function. This problem has been solved by Besikowitsch (Mathematische Zeitschrift, 1924) by the aid of a non-uniform function. The present author gives a uniform solution by means of a series of rational fractions, utilizing for this purpose one of his earlier results. (Received September 8, 1930.)

380.* Dr. A. A. Albert: *On algebras of type R in thirty-six units.*

The algebras considered by L. E. Dickson on pp. 51–79 of his *Algebren und ihre Zahlentheorie* are called algebras of type R. The author considers here algebras D of type R and order thirty-six over any non-modular field, where D is based upon a non-cyclic sextic, with regular group, and proves that every such algebra is a cyclic (Dickson) algebra. (Received August 11, 1930.)

381. Professor R. G. Smith: *A canonical form for the differential equations of curves in n-dimensional space.*

In this Bulletin (vol. 34, pp. 290–302) Professor E. B. Stouffer derived canonical forms for the linear homogeneous differential equations associated with curves in a plane or in ordinary space, and showed their geometrical significance. The canonical form presented in this paper is a generalization of the canonical form for curves in ordinary space. The results obtained are valid for any space of more than two dimensions. It is shown that the vertices of the canonical polyhedron of reference are uniquely fixed with respect to the curve. In particular, the $(n+1)$th vertex is in the unique linear $(n-1)$-space determined by the osculating linear $(n-2)$-space and the principal tangent plane (a generalization of the principal tangent plane as used in ordinary space) of the curve and its osculating curve of degree n. The canonical form leads at once to a complete system of invariants and covariants. (Received August 5, 1930.)

382. Dr. P. M. Swingle: *Two types of connected sets.*

It is shown that a bounded plane indecomposable continuum contains a connected subset C, such that every connected subset of C is dense in C. A number of theorems are proved concerning such connected sets and a related type. (Received August 25, 1930.)

383. Professor R. D. Carmichael: *Algebras of certain doubly transitive groups.*

A class of finite algebras $A[p^n]$ is defined directly by means of doubly transitive groups of prime-power degree p^n and order $p^n(p^n-1)$ and is shown to be equivalent to a class of finite algebras defined by Dickson in 1905 (Göttinger Nachrichten, 1905). The set of all linear transformations on the

* The papers beyond this point are to be read at meetings of which the reports have not yet been published.

marks of an $A[p^n]$ induces on those marks a group which is conjugate to that by which the algebra is defined. Three forms are given to the (only partially solved) problem of determining all algebras $A[p^n]$, one of them being of fundamental importance in the investigation of the group of isomorphisms of an Abelian group of order p^n and type $(1,1, \cdots, 1)$. This problem deserves further attention. Two algebras $A_1[p^n]$ and $A_2[p^n]$ are simply isomorphic when and only when their multiplicative groups are simply isomorphic. The integral elements of an algebra $A[p^n]$ form a Galois field $GF[p]$. The algebras $A[p^n]$ are capable of various analytical representations, including as a special case that employed by Dickson. A large class of doubly transitive groups of degree p^n and order $p^n(p^n-1)$ is exhibited and these groups are employed in the rapid construction of a large class of algebras $A[p^n]$ closely related to those determined by Dickson by other methods. (Received September 5, 1930.)

384. Professor R. D. Carmichael: *Tactical configurations of rank two.*

In accordance with the terminology of E. H. Moore, the term tactical configuration of rank two is employed to denote a combination of l elements into m sets, each set consisting of λ distinct elements and each element occurring in μ distinct sets (order of sets and order within a set being immaterial). Such configurations are important in the theory of permutation groups, in the formation of irrational invariants and in constructing poristic forms in connection with the study of geometrical configurations. In this paper numerous infinite classes of these tactical configurations are constructed by means of the finite geometries and by means of multiply transitive permutation groups. Several particular configurations of interest are also given. Means of constructing quadruple systems are set forth. The method of tactical configurations is employed in a new construction of the Mathieu groups of degrees 11, 12, 22, 23, 24. (Received September 5, 1930.)