THE APPLICATION OF MATHEMATICS TO THE SOCIAL SCIENCES*

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The invitation to deliver this lecture was accepted chiefly because of my veneration for J. Willard Gibbs, whose pupil I was forty years ago.

It was by accident rather than by design that my own life work was diverted from mathematical physics and mathematics, which it was my privilege to study under Gibbs, and was turned, instead, toward the application of mathematics to the social sciences.

While I have lost touch with the subject of which J. Willard Gibbs was a master, the debt which I owe to him and the help which my studies with him have afforded me even in the field of the social sciences can never be forgotten.

I hope, therefore, that it may not be amiss to precede what I have to say on Mathematics in the Social Sciences by a reminiscent statement of my personal impressions of Gibbs himself. J. Willard Gibbs towered, head and shoulders, above any other intellect with which I have come in contact. I had a keen realization of his greatness even in those formative years in Yale College and the Yale Graduate School. But this keen realization has grown even keener as the years have swept by, not only because of the increased evidence of the fundamental value of Gibbs' work in his own chosen field but also because in my own consciousness, after so many details have dropped from memory, there persists all the more clearly the strong impression which Gibbs' personality and teaching made upon me.

In saying this I do not think I can be accused of undue enthusiasm simply from the loyalty of a pupil to his teacher, especially in view of the statements of Lord Kelvin and others, which virtually rank Gibbs as the Sir Isaac Newton of America. Lord Kelvin said when visiting at Yale, a few years ago, that

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"by the year 2000 Yale would be best known to the world for having produced J. Willard Gibbs."

One of the most striking characterizations of Gibbs was recently made by Dr. John Johnston, now with the United States Steel Corporation, then Professor of Chemistry at Yale, in his address on Gibbs delivered at Yale University two years ago. He stated that no result of Gibbs' work had yet been overthrown, and that, in this respect, Gibbs seems to stand unique and supreme among the great scientists.

The English physical chemist, Professor F. G. Donnan, has said that "Gibbs ranks with men like Newton, Lagrange and Hamilton, who by the sheer force and power of their minds have produced those generalized statements of scientific law which mark epochs in the advance of exact knowledge . . . . . The work and inspiration of Gibbs have thus produced not only a great science but also an equally great practice. There is, today, no great chemical or metallurgical industry that does not depend, for the development and control of a great part of its operations, on an understanding and application of dynamic chemistry and the geometrical theory of heterogeneous equilibria."

Professor Ostwald said, in the preface to his German translation of Gibbs' thermodynamic papers in 1892:

"The importance of the thermodynamic papers of Willard Gibbs can best be indicated by the fact that in them is contained, explicitly or implicitly, a large part of the discoveries which have since been made by various investigators in the domain of chemical and physical equilibrium and which have led to so notable a development in this field . . . . . The contents of this work are today of immediate importance and by no means of merely historical value. For of the almost boundless wealth of results which it contains, or to which it points the way, only a small part has up to the present time 1892 been made fruitful."

Sir Joseph Larmor said:

"This monumental memoir *On the Equilibrium of Heterogeneous Substances* made a clean sweep of the subject and workers in the modern experimental science of physical chemistry have returned to it again and again to find their empirical principles forecasted in the light of pure theory and to derive fresh inspiration for new departures."

We think no less of Gibbs' greatness because he himself showed so little consciousness of it. He must have realized the fundamental character of his work. But his pupils remarked his profound modesty and often commented on it. His chief delight was in truth seeking for its own sake, and he was so intent on this search that he had no time even to think of emphasizing the originality or value of his own additions to the great vista of truth over which his mind swept. Doubtless he often did not know or greatly care where the work of others ceased and his own began. He did not always wade through the literature which preceded his own scientific papers. I remember hearing him say that when he wanted to verify another man's results he usually found it easier to work them out for himself than to follow the other man's own course of reasoning. This was said without any affectation, but simply in a jocular vein, as by one who would escape a difficult task by going his own way. But even though it may be difficult to disentangle what was original in Gibbs' work from what was anticipated by others, surely no competent person doubts today that he founded a new era in physics and chemistry.

Because of his retiring disposition and the theoretical nature of his work, he was, during his lifetime, almost unknown except among a few special students. The majority of students at Yale, in my day, did not know of his existence, much less of his greatness. And there were far fewer people in America who could appreciate what he was doing than there were in Europe.

His work was more promptly recognized in Germany. When I studied in Berlin in 1893, and was asked under whom I had studied in America, I enumerated the mathematicians at Yale. To my mortification not one of the names was known to those Berlin professors, until I mentioned Gibbs, whereupon they were loud in his praises. "*Geebs, Geebs, jawohl, ausgezeichnet!*"

Even today, Gibbs illustrates the rule that a prophet is not without honor save in his own country. As Professor Johnston noted, the fiftieth anniversary of the publication of the first part of Gibbs' great work on the Equilibrium of Heterogeneous Substances was signalized in Holland by the publication of a

Gibbs' number of their chemical journal. This contained contributions from Dutch authorities, as well as from French, German, Canadian, Norwegian and English authorities but none from an American!

It is true, however, that at Yale we have finally established a Gibbs fund for a lecturership to be filled by visiting professors and that a new complete edition of his works has been issued by Longmans, Green and Co. It is also planned to issue two volumes of commentations on Gibbs' work to make its chief results more accessible to the general scientist. I am proud to have played a part in these undertakings. May I take this opportunity to say that I am also proud to have been included among the J. Willard Gibbs lecturers? And may I congratulate the American Mathematical Society on being the organization to found this lecturership, although Gibbs was professedly not so much a mathematician as a physicist. The only other Gibbs lecturership seems to be that at the Mellon Institute in Pittsburgh. The Chicago Section of the American Chemical Society awards a "J. Willard Gibbs medal" annually, the recipient of which makes an address.

Presumably Gibbs' greatest contribution to science was his application of the laws of thermodynamics to chemistry. He made this almost a deductive science. Professor Bumstead said of his work: "To an unusual extent, among the sciences which appeal to experiment, it can be, and has been, cast in a deductive form. Sir Isaac Newton said that 'it is the glory of geometry that from a few principles it is able to produce so many things.' Thermodynamics shares in this kind of glory; it has only two fundamental principles, of which the first is a statement of the conservation of energy as applied to heat, and the second states the fact (so deeply founded in general experience that it seems almost axiomatic) that heat will not, of itself, flow from a body at a lower temperature to one of a higher temperature. From these two simple principles, by an almost euclidean method, a surprising number of facts and relations between work and heat, and various properties of bodies were deduced about the middle of the last century."

J. Willard Gibbs was certainly a master at producing many deductions from a few general principles. And it was just because of the generality of the principles from which he always insisted on starting that he succeeded in reaching such a wealth of conclusions.

In fact, it has always seemed to me that Gibbs' chief intellectual characteristic consisted in his tendency to make his reasoning as general as possible, to get the maximum of results from the minimum of hypotheses. I shall never forget, and have often quoted, an aphorism used by Gibbs, whether or not original with him, to the effect that "the whole is simpler than its parts." For instance, when he had a problem involving coordinates he preferred to employ an *indeterminate* origin, maintaining that his results were thereby rendered simpler and easier than if he took the origin at some apparently more convenient but special point in relation to the crystal or other conformation which he was discussing. When the origin is indeterminate it automatically effaces itself from all the general relations deduced.

Many, if not most, other investigators instinctively seek to solve special cases before attempting to solve the general case. Sometimes they pay a big penalty in needless experimentation. I remember Professor Bumstead, my fellow student at Yale, recounting with relish a conversation that Gibbs was reputed to have had with a youthful investigator who had made a laborious experimental study of certain relationships and who was, with pardonable pride, telling Gibbs of his conclusion. After listening attentively Professor Gibbs quite naturally and unaffectedly closed his eyes, thought a moment, and said, "Yes, that would be true", seeing at once that the special result which this young investigator had reached was a necessary corollary of Gibbs' own more general results. For him, it required no experimental verification. The young man's work had, from Gibb's viewpoint, been almost as much wasted as if it had been spent in a laborious set of measurements of right angle triangles on the basis of which measurements he should announce as a new and experimental discovery that the square of the hypotenuse is equal to the sum of the squares of the other two sides. It is worth noting that, though Gibbs did his work in the Sloane Physics Laboratory, he never, as far as I know at least, performed a single experiment. His life work, stupendous as it was, and based as it was on concrete fact, consisted exclusively in new deductions from old results, the full significance of which no one else had been able to derive.

In his effort to represent physical relations by geometrical models and to portray the theory of electricity and magnetism by geometrical methods, Gibbs encountered the need of a new vector analysis to replace the awkward analysis by Cartesian coordinates, requiring, as that does, three times as many equations to write and manipulate as does vector analysis, not to say diverting attention from the lines and surfaces actually involved to their projections on three arbitrary axes.

To me the most interesting course I had with Gibbs was on vector analysis. He believed he had simplified the Hamilton system of quaternions, getting his cue from Grassmann's *Ausdehnungslehre.* But he was so conscious of his obligations to Grassmann that he was reluctant to publish his own system, apparently doubting whether it possessed enough originality to warrant publication. He therefore had privately printed a syllabus of his system and this reprint was used by us in his class as a text. Only after many years did Professor E. B. Wilson construct a more elaborate text book embodying Gibbs' principles of vector analysis.

It is a curious fact that, while Gibbs' work in thermodynamics was appreciated in Germany, his work in vector analysis was not. I remember the comment of Professor Schwartz at Berlin, when I undertook to defend Gibbs' vector analysis: "*Es ist zu willkürlich.*" The Germans felt in honor bound to restrict pure mathematics to mere elaboration of the proposition that one and one make two. Starting with this proposition, by successive additions or subtractions of unity, we may, of course, by going forward, obtain all the positive integers; this is addition. Reversing, we obtain zero and the negative integers; this is subtraction. Then, by applying repeated addition or multiplication and repeated subtraction or division, repeated multiplication or involution, and repeated division or evolution, we arrive at fractions, surds, imaginary quantities and finally the *complex variable* $x+yi$. But beyond this, by such processes no more general form of magnitude can possibly be derived; for, if we operate on a complex variable by addition, subtraction, multiplication, division, involution or evolution, under the recognized rules of algebra, we obtain simply other complex variables and nothing else whatsoever. Only by changing, or as the German critics would say, by violating, the fundamental rules of algebra faithfully followed in the above processes, such as the rule that $a \times b$ is equal to $b \times a$, is it possible to enter into any other realm of mathematics than that of the complex variable.

When I reported these criticisms to Gibbs, his comment was that all depends on what your object is in making those sacrosanct rules for operating upon symbols. If the object is to interpret physical phenomena and if we find we can do better by having a rule that $a \times b$ is equal not to $b \times a$ but to minus $b \times a$, as in the multiplication of two vectors, then, he said, the criticisms of the Germans are beside the point.

The fact is that Gibbs, though a great mathematician, was not primarily interested in mathematics as such. His interest lay in its applications to reality—in the substance rather than the form. All his contributions to pure mathematics were sought and found not as mere proliferations of formal and abstract logic but as by-products of his work in interpreting the facts of the physical universe.

The far reaching effects of Gibbs' work apply not only to inorganic physics and chemistry but also to the organic world. One of the most elaborate reviews of Gibbs and his relation to modern science is by Lt. Col. Fielding H. Garrison, M.D., Assistant to the Librarian of the Army Medical Library, Washington, D. C., in which he shows, among other things, the application of Gibbs' work to the equilibrium of heterogeneous substances in general physiology.

Despite Gibbs' retiring disposition and his disinclination for general society he was most cordial in his personal contact with colleagues and students and never seemed to lack time to give to anyone who chose to discuss the subject in which he was so deeply interested. He made on all a deep impression of kindliness. I well remember the remark of Percey Smith, now Professor of Mathematics at Yale, who was a fellow student of Gibbs with me, as we walked out of one of Gibbs' lectures, "What a *gentle* man he is!"

He enjoyed a joke, often laughed and excited laughter. He took pleasure in applying his mathematics in simple ways. One of his minor but fascinating papers before the Yale Mathematical Club was on the *Paces of a horse*, the writing of which was doubtless suggested by watching a horse which he had just purchased. Probably no one else ever put a horse through his paces as scientifically or amusingly as Gibbs did in that paper.

Gibbs himself never contributed to the social sciences. Apparently I am the only one of his pupils who, after first doing some teaching in mathematics and physics, became professedly an economist, although Professor E. B. Wilson, Gibbs' chief interpreter as to mathematics, has taken a lively interest in many lines of social science and statistics and was this year President of the American Statistical Association.

After several years' graduate study partly in mathematics under Gibbs, and partly in economics under Sumner, the time came for me to write my doctor's thesis and I selected as my subject *Mathematical Investigations in the Theory of Value and Prices*. Professor Gibbs showed a lively interest in this youthful work, and was especially interested in the fact that I had used geometric constructions and methods, including his own vector notation.

The late Professor Allyn Young of Harvard also made occasional use of vectors in his economic work. Another economic student and writer, a brilliant young Norwegian, Professor Ragnar Frisch, has latterly used the vector notation and says he could scarcely think without it. Professor Frisch will this year be visiting Professor at Yale from the University of Oslo.

It is one of the handicaps of mathematics in the social sciences that there are so few who are trained in both lines for such study, and this particularly applies to any applications of Professor Gibbs' vector analysis. If vector analysis should become more widely understood and used by students in the social sciences, doubtless it would be more generally utilized, at least as a vehicle for thought.

Occasionally, and increasingly, the ideas and notations of the differential and integral calculus are applied by mathematical economists and statisticians. But, of course, most of the mathematics employed in the social sciences consists of simple algebra. There is a saying which, by the way, was quoted by Gibbs in his address on Multiple Algebra, that "the human mind has never invented a labor-saving machine equal to algebra."

There are several fairly distinct branches of social science to which mathematics has been, or may be, applied. The chief of these may be distinguished as (1) pure economics, (2) the "smoothing" of statistical series, (3) correlation, and (4) probabilities, all of which overlap to some extent.

My own chief interest in social science, from a mathematical point of view, has been in the first of these four groups, pure theory.

When I began my work in this field in 1891, mathematics in economic theory was looked at askance, despite the fact that many years before, as early as 1838, Cournot had written his brilliant *Researches into the Mathematical Principles of the Theory of Wealth.* This book later greatly stimulated Professor Edgeworth of Oxford and Professor Marshall of Cambridge, and today is ranked among the economic classics. The same may be said of Jevons' *Theory of Political Economy,* published in 1871. But in 1891, when my own economic studies began, even the work of Cournot was almost unknown to economists, and that of Jevons was little used. If one will turn the pages of the main economic literature of 1891 and earlier, he will find practically no formulas and no diagrams. But Walras and Pareto in Switzerland and Pantaleoni and Baroni in Italy, Edgeworth and Marshall in England, Westergaard and Wicksell in Scandinavia, and a few other students in other countries were using and defending the new method.

When, at the request of Professor Edgeworth, I read a slightly mathematical paper on the *Mechanics of bimetalism* before the Economic Section of the British Association for the Advancement of Science at Oxford in September, 1893, I well remember how, in the discussion of that and other mathematical papers, Professor Edgeworth was, as he expressed it, "damped" by the unfriendly criticism of these new methods by Professor Sidgwick and others.

But little by little, the usefulness of mathematics has come to be appreciated. Besides the older writers already mentioned and Auspitz and Lieben, whose work on price determination of 1889 was one of my first inspirations, there have gradually come into this field many younger writers, among whom may be mentioned Professor Henry L. Moore of Columbia University, Professor J. H. Rogers of the University of Missouri, Professor C. F. Roos of Cornell University, Professor Griffith C.

Evans of Rice Institute, Professor Henry Schultz of the University of Chicago, Professor Harold Hotelling of Stanford University, W. A. Shewhart of the Bell Telephone Laboratories, Professors J. Maynard Keynes, A. C. Pigou, and Arthur L. Bowley of England, Professors Albert Aftalion and Jacques Rueff of France, Professor L. von Bortkiewicz and Dr. Otto Kühne of Germany, Professor Wl. Zawadski of Poland, Professor E. Sluts Ky of Russia, Professor Gustav Cassel of Sweden, Professor Ragnar Frisch of Norway, Dr. Willem L. Valk of Holland, Professors Corrado Gini and Luigi Amoroso of Italy.

And, besides the fact of such accessions to the ranks of the small band of professed mathematical economists, is the even more significant fact that economists in general have not only ceased decrying mathematics but are, in many cases, making some slight use of it themselves.

The late Professor Marshall of Cambridge University was one of the first to perceive what was happening. He said:

"A great change in the manner of thought has been brought about during the present generation by the general adoption of semi-mathematical language for expressing the relation between small increments of a commodity on the one hand, and on the other hand small increments in the aggregate price that will be paid for it: and by formally describing these small increments of price as measuring corresponding small increments of pleasure. The former, and by far the more important, step was taken by Cournot (*Recherches sur les Principes Mathématiques de la Théorie des Richesses*, 1838); the latter by Dupuit (*De la mésure d'utilité des travaux publics*, in the Annales des Ponts et Chaussées, 1844), and by Gossen (*Entwickelung der Gesetze des menschlichen Verkehrs*, 1854). But their work was forgotten; part of it was done over again, developed and published almost simultaneously by Jevons and by Carl Menger in 1871, and by Walras a little later. Jevons almost at once arrested public attention by his brilliant lucidity and interesting style. . . . .

"A training in mathematics is helpful by giving command over a marvellously terse and exact language for expressing clearly some general relations and some short processes of economic reasoning; which can indeed be expressed in ordinary language, but not with equal sharpness of outline. And, what is of far greater importance, experience in handling physical problems by mathematical methods gives a grasp, that cannot be obtained equally well in any other way, of the mutual interaction of economic changes. The direct application of mathematical reasoning to the discovery of economic truths has recently rendered great services in the hands of master mathematicians to the study of statistical averages and probabilities and in measuring the degree of consilience between correlated statistical tables."

Mathematics serves economic theory in supplying such fundamental concepts based on the differential calculus and also through the process of differentiation solves problems of maxima and minima, as in the simple process of determining formally what is the price that the traffic will bear in order to make profits a maximum.

The chief realm of economic theory to which mathematical analysis of this formal kind applies is that of supply and demand, the determination of prices, the theoretical effect of taxes or tariffs on prices. The results cannot always be reduced to figures but are often useful in terms of mere inequalities.

For instance, among the chief theorems shown mathematically by Cournot are the following:

That a tax on a monopolized article will always raise its price, but sometimes by more and sometimes by less than the tax itself.

That a tax on an article under unlimited competition always raises its price but by an amount less than the tax itself.

That a tax proportional to the net income of a producer will not affect the price of his product.

That fixed charges among costs of production do not affect price nor do taxes on fixed charges.

That opening up free trade in a competitive article between two previously independent markets may decrease the total product.

Among the most surprising paradoxes discovered by the mathematical method is one shown by Edgeworth, that if a monopolist sells two articles, say first and third class railway tickets for which the demand is correlated, it may be possible to tax the third class tickets, at a fixed amount each, with the result that the monopolist not only pays the tax but lowers the prices of both kinds of tickets.

Familiarity with mathematics will save many confusions of thought. For instance, it is just as important in economics to distinguish between a *high* price and a *rising* price as it is in physics to distinguish between velocity and acceleration. *Rate of price change* has important effects, both theoretically and in practice, on the rate of interest and on the volume of business.

Theoretically the rate of interest ought to be higher during a period of rising prices, or depreciation of the dollar, by an amount equal to the rate of depreciation and it ought to be lower during appreciation.

Practically, however, the rate of interest is slow of adjustment and what is more important, inadequate in adjustment. A mathematical statistical analysis of this slowness and inadequacy helps explain great business upheavals as shown in my new book on *The Theory of Interest.* I may say here, parenthetically, though the case is somewhat different, that the recent crash in the stock market was, in large measure, the price paid for tardiness in raising the rate of interest which should have risen over a year ago but was held down artificially.

Again, mathematics will save the economic student and the student of accounting from the many confusions of double counting, especially in the intricate theory of income.

Another elementary, but important, use of mathematics in economics is in making sure that a problem is determinate by counting and matching the number of independent equations and the number of unknown quantities. A great deal of unnecessary misunderstanding has existed and still exists in economic science as to what determines the rate of interest or other magnitudes in economics. These misunderstandings would not exist if the contestants would take the trouble to express themselves mathematically. If we view the matter mathematically it soon becomes evident that one contestant has seen only one of the determining factors, and the other another, without either of them realizing that both are compatible and needed in a complete economic equilibrium. The concept of economic equilibrium in which many factors act and react on each other is one of the chief elementary contributions of mathematics to economic theory, and one stressed by Cournot, Walras, Marshall, Pareto and Edgeworth.

Still another use of mathematics is in illustrating geometrically or analytically the fact that a price, or a *marginal utility*, is a function not simply of one but of many variables, the function being purely empirical and incapable of analytical or arithmetical expression. In fact, the economic world is a world of n dimensions.

Thus the marginal utility of bread to John Doe is a function of his quantity not only of bread consumed, but of butter, sugar, and numerous other variables.

I have myself tried to apply these and other mathematical ideas to the formal solution of the problem of prices of commodities, the rate of interest, the relation of capital to income, the purchasing power of money, and what Simon Newcomb, the astronomer-economist, called the equation of societary circulation, now called the equation of exchange (the volume of circulating medium multiplied by its velocity of circulation is equal to the price level multiplied by the volume of trade per unit of time).

Most of these and other applications of mathematics to economic theory consist in short chains of reasoning. Professor Marshall had the impression that only short chains of reasoning could ever be expected in mathematical economics. He said: "It is obvious that there is no room in economics for long trains of deductive reasoning: no economist, not even Ricardo, attempted them. It may indeed appear at first sight that the contrary is suggested by the frequent use of mathematical formulas in economic studies. But on investigation it will be found that this suggestion is illusory, except perhaps when a pure mathematician uses economic hypotheses for the purpose of mathematical diversions; for then his concern is to show the potentialities of mathematical methods on the supposition that material appropriate to their use had been supplied by economic study. He takes no technical responsibility for the material and is often unaware how inadequate the material is to bear the strains of his powerful machinery."

But as time goes on, there appear instances of somewhat longer trains of reasoning.

I may take an example from my own work. I have tried to show how it is possible to estimate numerically, through suitable mathematical equations, the velocity of the circulation of money. The formula for this was derived through a chain of mathematical reasoning requiring several links and embracing a considerable number of variables of which the chief are the volume of money in circulation, the annual flow of money into and out of banks and the annual cash payments to labor. This problem, by the way, of evaluating the velocity of circulation of money had been pronounced insolvable and, without mathematical analysis, it might well be so considered. Incidentally, the calculations indicate that money in the United States circulates about 25 times a year. In other words, the average dollar stays in the same pocket about two weeks.

To turn to a different example, both Professor Ragnar Frisch and myself, by independent methods, both of them highly mathematical, have shown how, theoretically under certain simple hypotheses, to accomplish the statistical measurement of "marginal utility" or desirability, as a function of one's income. This would determine whether or not it is true that if one man has double the income of another his tax ought to be double, more than double, or less than double in order that he should make the same sacrifice. In other words, it would supply a mathematical criterion by which to judge the justice of a progressive income tax.

I say "would" rather than "does" simply because as yet the statistics available do not seem adequate for any accurate evaluation. But Professor Frisch and I are both hoping to pursue this study further. His and my preliminary results are not inconsistent. My own formula is derived by a chain of mathematical reasoning which results in expressing the ratio of the "marginal utility" of money for a person with a certain income to the "marginal utility" which he would have with a different income in terms of the following elements: those two incomes, the percentages which would be spent on food, rent, etc., under the two respective incomes and the index numbers of prices of food, rent, etc., relatively to another country, serving as a standard of comparison.

Mathematics also helps make clear the "dimensionality" of the magnitudes treated. Thus, the quantity of wheat, its price and its value are three magnitudes as unlike in dimensionality as time, velocity and distance. The rate of interest has the simplest dimensionality, being, like angular velocity, of dimension t^{-1}.

Mathematics helps us analyze time relationships in general, especially to avoid the old confusion between capital and income, the one relating to an instant of time, the other to a period of time.

Capital-income analysis is a development of the last two score years; but its roots go back generations. Every good treatise on algebra includes the formulas for capitalizing annuities and bonds, the formulas underlying the bond tables used in every broker's office.

While the development of mathematical economics from the theoretical side has been steady and impressive since I was a young man, it has by no means been so rapid as the development of the other three branches to which I have referred.

"Smoothing" statistical data, the fitting of formulas and curves to statistics, has, of course, been one of the aims of statisticians for many generations. In this way we have derived our mortality tables, the basis used by actuaries for calculating life insurance premiums.

I understand that the second J. Willard Gibbs lecture was by Robert Henderson on *Life Insurance as a Social Science and as a Mathematical Problem*. The importance of this branch of our subject does not need to be emphasized in an insurance center like DesMoines.

Actuarial science is, of course, a development of the formulas for capitalization or discount, particularly as applied to annuities, combined with the introduction of the probability element based on mortality statistics. It is essentially an analysis of interest and risk. It could be, and perhaps some day will be, applied to other economic problems besides life insurance, as soon as statistics are adequate for assessing risk numerically in other realms than human mortality. In fact one of the crying needs of economic science is a reliable basis for evaluating risks.

Concurrently with actuarial science has developed a science of mathematics of mortality in relation to population, extending at least back to the days of William Farr, Superintendent of the statistical department of the Registrar General's office of England half a century ago. Today this science has been further developed by Knibbs of Australia, Lotka, and Glover in the United States, and others.

Recently, with the development of statistics of industry, the art of curve fitting, by mathematical methods, has grown very rapidly, and examples of it will be found in many current issues of statistical journals. I am, myself, with a collaborator, Max Sasuly, working on a new method of curve fitting aimed to avoid the use of any preconceived formula but letting the statistical data themselves write their own formula, so to speak.

One important phase of curve fitting which links it closely with the study of economic theory is the statistical evaluation of supply and demand curves. Among those who have worked in this field are Professor Henry L. Moore of Columbia University, Professor Henry Schultz of Chicago University, Dr. Mordecai Ezekiel of the United States Department of Agriculture, Professors G. F. Warren, F. A. Pearson and C. F. Roos of Cornell University, and Professor Holbrook Working of Stanford University. Professor Schultz was apparently the first to work out the statistical determination of the effect of the tariff on the price of an imported commodity—sugar.

The third group of mathematical work in the social sciences, the development of correlation, is closely associated with the name of Karl Pearson of the University of London, who is still living. His "correlation coefficient" has become almost a standard procedure in economic statistics as well as in other sciences, including biology, in which he is primarily engaged. Today many, if not most, economists, especially if they work in statistics, are making some use of such correlation coefficients. Through them they have been forced to adopt mathematical aids in spite of old traditions and prejudices.

Professor Warren M. Persons, formerly of Harvard, has worked out correlations with lag showing the interrelations of various economic phenomena in such a way as to serve the purposes of forecasting business conditions. A more elaborate method of correlation has been worked out by Karl Karsten of New Haven, a private statistician, who has made tables of correlation between every pair of available series of economic statistics and has put these together by multiple correlation so as to predict any one of the series from all of the others which are found to serve toward that end. He is now issuing regularly a forecast of commodity prices, of the volume of business, of stock market trends, and of various other economic factors.

In the study of the so-called "business cycle" and forecasting future fluctuations, mathematical economists and statisticians have made increasing use of what is virtually differentiation or integration. Thus I have emphasized "price-change" as distinct from price, of which it is the differential quotient. Reciprocally Mr. Karsten has applied the idea of "quadrature" to the relations of two statistical series where one is virtually derivable from the other by integration. This means if the curves are cyclical that they are related as are the curves of sines and cosines so that one curve is at zero while the other is at a maximum or minimum.

One development in this field in which I have been especially interested has been the study of the distribution of the lag. This appears to be a skew distribution, but nearly normal if time is plotted on a logarithmic scale.

As already indicated, risk is one of the fundamental elements in the mathematical analysis of actuarial science. It is also, in a different way, through frequency distribution, a fundamental element in correlation analysis. In fact, there have been more or less successful attempts by Karl Pearson to resolve a mortality curve into a sum of several frequency curves and by Arne Fisher to do the reverse, synthesize a set of frequency curves representing mortality from certain causes into the total mortality curve. It may also be pointed out that our second topic, curve fitting and smoothing, whether by least square methods or otherwise, is largely a study in probability.

All this brings us to the fourth chief branch of our subject, the mathematical analysis of probability in general insofar as this relates to social phenomena as embodied in statistics. This has been increasingly studied by many economists, especially the late F. Y. Edgeworth, editor of the Economic Journal. Also mathematical statisticians such as G. Udny Yule, Arthur Bowley, R. A. Fisher, Sir William Beveridge, Truman L. Kelly, A. C. Whitaker, William L. Crum, Thiele and others have done much constructive work in this field.

Professor Vilfredo Pareto tried to work out a formula for statistics of incomes in relation to the number of persons possessing incomes of various sizes, and the Pareto curve has become quite famous. It has been shown, however, particularly by Professor Macaulay of the National Bureau of Economic Research, that the Pareto curve is nothing but the "tail" of a probability curve, although Pareto himself had been loath to admit this. It is true that this particular sort of probability or distribution curve is not "normal" even if the abscissas are plotted on a logarithmic scale. It often happens in statistical series, especially where the frequency distribution lies between zero as one extreme and infinity as the other, that the frequency or probability curve while very skew on the arithmetical scale turns out to be nearly "normal" on the logarithmic scale.

I have, of course, by no means exhausted the list of applications of mathematics to economics, still less to the other social sciences. Many applications have been made which are not easily classified under the four heads I have noted, namely, pure theory, curve fitting, correlation and probabilities.

Of these other and miscellaneous applications, the most important, at least in the field of economics and statistics, seems to be that of index numbers. The theory and practice of index numbers have had a special fascination for many of us because they occupy a tantalizing position on the borderline between a priori rational theory and empirical makeshift and because of the many pitfalls encountered in their study. It is closely related to the subject of probability. In my book on *The Making of Index Numbers*, I have tried to find the best formula for an index number out of several score available formulas.

It is also true, of course, that the last three divisions of our subject, curve fitting, correlation study, and probability, traverse all fields of knowledge. They apply not only to my own branch of the social sciences, economics, but to all others such as sociology, anthropology, and education, as well as to fields outside of social science such as psychology, biology, hygiene and eugenics. In all of these we find a development of mathematical method. Each has its own special concepts, measures, and principles such as the cranial index of anthropology, the intelligence quotient of psychology and education, the Mendelian principle in heredity, and these the mathematician may study in terms of averages, index numbers, correlations, deviations, frequency distributions and otherwise. Just as the multiplication table is applicable in more than one field of knowledge, so mathematics in general is peculiar to none. Sooner or later every true science tends to become mathematical. The social sciences are simply a little later to be reached than astronomy, physics and chemistry, while the biological sciences are later still.

Scientific method is one and the same, whether employed in such sciences as Gibbs developed, or in others. Mathematical notation is, as Gibbs said, simply a *language*. It is required for the best expression of scientific method when the relations to be expressed become sufficiently involved to require it in preference to ordinary language which is less precise and complete. The outlook is bright for a healthy development of mathematics in the social sciences.

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