

A PARTIAL ISOMORPHISM BETWEEN THE FUNCTIONS OF LUCAS AND WEIERSTRASS*

BY E. T. BELL

1. *Introduction.* In a former paper† it was pointed out that certain identities in the theory of multiplication, real or complex, of elliptic functions, are of precisely the same form as others between Lucas' U_n, V_n . The latter were defined only for integer values of n by Lucas as follows,

$$U_n \equiv \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n \equiv \alpha^n + \beta^n,$$

where α, β are real or complex numbers such that $\alpha + \beta = p$, $\alpha\beta = q$, where p, q are rational numbers. Hence U_n, V_n are rational numbers and are solutions of

$$W_{n+2} = pW_{n+1} - qW_n,$$

with the initial conditions

$$(U_0, U_1) = (0, 1), \quad (V_0, V_1) = (2, p).$$

It follows, as shown by Lucas,‡ that U_n, V_n are simply expressible as circular functions of a real or imaginary argument, according as α/β is imaginary or real. Thus every formula in elliptic functions will contain as a degenerate case one concerning the U_n, V_n functions, but this is not the isomorphism sought.

We first generalize Lucas' definition in one respect and specialize it in another, replacing n by the complex variable z and restricting q to be 1. We shall write the functions thus defined as

$$(1) \quad U(z) \equiv \frac{\alpha^z - \alpha^{-z}}{\alpha - \alpha^{-1}}, \quad V(z) \equiv \alpha^z + \alpha^{-z},$$

* Presented to the Society, December 31, 1928.

† This Bulletin, vol. 29 (1923), pp. 401-406.

‡ American Journal of Mathematics, vol. 1, (1878), p. 189.

where α, p are constants, real or complex, such that $\alpha + \alpha^{-1} = p$. The principal determinations of exponentials and logarithms are to be understood. From $U(z), V(z)$ we shall construct certain functions exhibiting a partial isomorphism with the $\sigma(z), \zeta(z), \wp(z)$ of Weierstrass. A complete isomorphism is obviously impossible, $U(z), V(z)$ being only singly periodic.

To make the sense in which we are using isomorphism precise, we recall a well known theorem for the sigma function.* Let $f'(z)$ denote the derivative of $f(z)$ with respect to z , and suppose that $f(z)$ satisfies the following conditions.

(A) $f(z)$ is a transcendental integral function of z .

(B) $f'(0) = 1$.

(C) $f(z+a)f(z-a)f(b+c)f(b-c)$
 $+ f(z+b)f(z-b)f(c+a)f(c-a)$
 $+ f(z+c)f(z-c)f(a+b)f(a-b) = 0,$

where z, a, b, c are independent complex variables. Then necessarily $f(z) \equiv \sigma(z)$. If now $f(z)$ satisfies (A) and (C), but not (B), we shall call $f(z)$ a *partial isomorph* of $\sigma(z)$. As shown by Kronecker,† the “three-term relation” (C) is formally equivalent to Jacobi’s formula for the multiplication of four elliptic theta functions. Hence, if preferred, Jacobi’s relation may replace (C) in what follows. The notation $\text{sh } x, \text{ch } x$ is used for the hyperbolic sine and cosine of x .

2. *The Functions* λ, μ, ν . With α, p as in (1), define the constants k, g by

$$(2) \quad k \equiv \log \alpha, \quad 2g \equiv k \text{ hs } k,$$

where $\text{hs } x \equiv 1/\text{sh } x$. Then $p = 2 \text{ ch } k$, and

$$W(z+2) - 2 \text{ ch } kW(z+1) + W(z) = 0$$

has the solutions $W(z) \equiv U(z), V(z)$.

* Halphen, *Traité des Fonctions Elliptiques*, vol. 1, p. 187.

† Journal für Mathematik, vol. 102, p. 260.

Write $\text{ch } x/\text{sh } x \equiv \text{cs } x$, and define the functions λ, μ, ν by

$$(3) \quad \lambda(z) \equiv \frac{1}{2g}U(z) = \frac{1}{k} \text{sh } kz,$$

$$(4) \quad \mu(z) \equiv \frac{g^2V^2(z)}{U^2(z)} - k^2 = k^2 \text{hs } kz,$$

$$(5) \quad \nu(z) \equiv g \frac{V(z)}{U(z)} = k \text{cs } kz.$$

The hyperbolic forms can be verified by inspection from (2) and

$$(6) \quad U(z) = \frac{2g}{k} \text{sh } kz, \quad V(z) = 2 \text{ch } kz.$$

3. *Correspondence between λ, μ, ν and σ, \wp, ζ .* We shall now show that $\lambda(z), \mu(z), \nu(z)$ correspond, up to a certain point, to $\sigma(z), \wp(z), \zeta(z)$ respectively. Note first that $\lambda'(0) = 1$, so that §1 (B) does not hold. The addition and subtraction theorems for $U(z)$ are

$$2U(z \pm w) = U(z)V(w) \pm V(z)U(w),$$

as is evident from (6), and therefore, if $K \neq 0$ and L are arbitrary constants, and if for the moment we write $W(z) \equiv KU(z)$, we have the identity

$$\begin{aligned} & \left[\left\{ \frac{V(z)}{2W(z)} \right\}^2 + L \right] - \left[\left\{ \frac{V(w)}{2W(w)} \right\}^2 + L \right] \\ &= - \frac{W(z+w)W(z-w)}{W^2(z)W^2(w)}. \end{aligned}$$

The correspondence will be effected by properly choosing K, L , as the last identity is of the same form as

$$(D) \quad \wp(z) - \wp(w) = - \frac{\sigma(z+w)\sigma(z-w)}{\sigma^2(z)\sigma^2(w)},$$

from which the three-term relation for σ follows at once, in the usual manner, by means of the algebraic identity

$$(b - c)(d - a) + (c - a)(d - b) + (a - b)(d - c) \equiv 0.$$

The constants are determined by assuming $W(z)$ to be the correspondent of $\sigma(z)$, and recalling the definitions

$$(7) \quad \zeta(z) \equiv \sigma'(z)/\sigma(z), \quad \wp(z) = -\zeta'(z).$$

A simple calculation gives the functions $\lambda(z)$, $\mu(z)$, $\nu(z)$ as defined in §2, (3)–(5). Hence, or directly from (3)–(5), we see that

(8) $\lambda(z)$ satisfies §1 (A), (C), but not (B), so that $\lambda(z)$ is a partial isomorph of $\sigma(z)$;

$$(9) \quad \nu(z) = \lambda'(z)/\lambda(z), \quad \mu(z) = -\nu'(z),$$

corresponding to (7);

$$(10) \quad \lambda(-z) = -\lambda(z), \quad \nu(-z) = -\nu(z), \quad \mu(-z) = \mu(z),$$

corresponding to the like for $\sigma(z)$, $\zeta(z)$, $\wp(z)$ respectively;

$$(11) \quad \mu'^2(z) = 4\mu^3(z) + 4k^2\mu^2(z);$$

and hence, if $\theta(z) \equiv \mu(z) + k^2/3$,

$$(12) \quad \theta'^2(z) = 4\theta^3(z) - k_1\theta(z) - k_2,$$

corresponding to the Weierstrass normal form with the invariants $k_1 = 4k^4/3$, $k_2 = (-2k^2/3)^3$.

The consequences of this correspondence can be developed indefinitely, without further computations. For example, since the correspondent of (D) is

$$(13) \quad \mu(z) - \mu(w) = -\frac{\lambda(z+w)\lambda(z-w)}{\lambda^2(z)\lambda^2(w)},$$

we infer the following correspondent for the addition theorem for the \wp function in one of its usual forms,

$$\mu(z \pm w) = \mu(z) - \frac{1}{2} \frac{d}{dz} \left[\frac{\mu'(z) \mp \mu'(w)}{\mu(z) - \mu(w)} \right],$$

and the like with θ in place of μ . Again, since the limit of $\lambda(t)/t$ as t approaches 0 is 1, we divide (13) throughout by

$z-w$, take the limit as w approaches z , and find $\mu'(z) = -\lambda(2z)/\lambda^4(z)$, the correspondent of $\varphi'(z) = -\sigma(2z)/\sigma^4(z)$.

If we wish to interpret this partial isomorphism in terms of Lucas' U_n, V_n , where n is an integer, we may so do in the special case (cf. §1) in which $q=1$. The derivatives $\lambda'(z)$, $\mu'(z)$, \dots are then replaced by their equivalents in terms of $U(z), V(z)$ with z finally replaced by n .

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AN INEQUALITY FOR DEFINITE HERMITIAN DETERMINANTS*

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The proof of M. Ragnar Frisch's theorem,† *The absolute value of a symmetric, definite determinant of real elements is at most equal to the product of the absolute values of the elements of the principal diagonal*, may be generalized to establish the following theorem of which the above is clearly a special case: *The absolute value of a definite Hermitian determinant is at most equal to the product of the absolute values of the elements of the principal diagonal.*

An Hermitian determinant

$$(1) \quad H \equiv \begin{vmatrix} h_{11} & \cdots & h_{1n} \\ \cdots & \cdots & \cdots \\ h_{n1} & \cdots & h_{nn} \end{vmatrix}$$

is a determinant such that

$$h_{rs} = \bar{h}_{sr}, \quad (r, s = 1, 2, \dots, n),$$

* Presented to the Society, February 23, 1929.

† *Sur le théorème des déterminants de M. Hadamard*, Comptes Rendus, vol. 185 (1927), p. 124. This is not a new theorem; see Bachmann, *Die Arithmetik der Quadratischen Formen*, 1923, pp. 250–251. It is more the method of proof than the result that makes Frisch's paper of interest. Though Bachmann does not give the generalized proof of the present paper, his proof holds equally for it.