APPROXIMATE SOLUTIONS OF CERTAIN GENERAL TYPES OF BOUNDARY PROBLEMS FROM THE STAND-POINT OF INTEGRAL EQUATIONS*

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1. *Introduction.* The principle of Rayleigh†, which is used by physicists to obtain practical solutions of certain boundary problems, assumes that the system under consideration may be replaced by an approximating algebraic system so chosen that the difference between the solutions of the two systems is negligible. It is evident that the practical value of this principle depends not only on the possibility of choosing an approximating system which can be easily solved, but also on the possibility of estimating the order of difference between the solutions.

In a paper published in 1923, M. Plancherel‡ presented a justification of the use of the difference system as an approximating system for a second order self adjoint linear differential equation with simple boundary conditions. He does not discuss the order of approximation of solutions. Such a discussion has been given by N. Bogoliouboff and N. Kryloff.§ R. Courant,|| in a paper which appeared in 1926, showed that certain integro-differential boundary problems could also be approximated by the method of

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† Lord Rayleigh, *Theory of Sound*, vol. 1, pp. 89–96.

‡ M. Plancherel, Bulletin des Sciences Mathématiques, (2), vol. 47 (1923), pp. 153-160, 170–177.

§ N. Bogoliouboff and N. Kryloff, Annals of Mathematics, (2), vol. 29 (1928), pp. 255–275.

|| R. Courant, Acta Mathematica, vol. 49 (1926), pp. 1-67.

difference equations. In this paper we use an essentially different method.* The class of problems to which it can be applied is not restricted to the class of problems that originated in the calculus of variations, with which Courant has been concerned. In some respects, however, the method used by Courant is more general since it can be applied to non-linear as well as to partial differential boundary problems.

2. *Preliminary Theorems: Approximate Solutions of the Fredholm Integral Equation.*† We obtain for reference in this and later sections an estimate of the differences between the characteristic numbers of the Fredholm integral equation

$$(A_1)\qquad u(x) + \lambda \int_a^b k(x, s)u(s)ds = f(x),$$

and those of

$$(A_2)\qquad u(x,\epsilon) + \lambda \int_a^b k(x,s,\epsilon)u(s,\epsilon)ds = f(x,\epsilon),$$

where, in the region $a \leqq {}_s^x \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$, the functions $k(x, s, \epsilon)$ and $k(x, s)$ are bounded and integrable (in the Lebesgue sense) and

$$k(x,s,\epsilon) - k(x,s) = O(\epsilon)\ ;\quad f(x,\epsilon) - f(x) = O(\epsilon).‡$$

If ϵ_0 is sufficiently small, then

$$k(x,s,\epsilon) - k(x,s) = \epsilon h(x,s,\epsilon)$$

where $h(x, s, \epsilon)$ is uniformly bounded in the region $a \leqq {}_s^x \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$, and $f(x, \epsilon) - f(x) = \epsilon h(x, \epsilon)$, where $h(x, \epsilon)$ is uniformly

* This method was suggested by Professor J. D. Tamarkin to whom the author is indebted for advice and criticism during the preparation of this paper.

† E. Schmidt, Mathematische Annalen, vol. 63, pp. 467–472, and vol. 64, pp. 161–174; H. Bateman, Proceedings of the Royal Society, A, vol. 100, pp. 441–449; F. Tricomi, R. Accademia dei Lincei, vol. 33, sem. 1, pp. 483–486, and vol. 33, sem. 2, pp. 26–30; R. B. Adams, Thesis in Candidacy for the degree of Doctor of Philosophy submitted to Radcliffe College, 1921; H. Block, Lunds Universitets Årsskrift, N. F. Afd. 2, 7, no. 1, (1911); G. C. Evans, American Mathematical Monthly, vol. 34, pp. 148–150.

‡ By definition $f(\epsilon) = O(\epsilon^\alpha)$ if $f(\epsilon)/\epsilon^\alpha$ is bounded.

bounded in the region $a \leqq x \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$. The integral equation

$$(1) \qquad u(x,\rho,\epsilon) + \lambda \int_a^b k(x,s,\rho,\epsilon)u(s,\rho,\epsilon)ds = f(x,\rho,\epsilon)$$

in which

$$k(x,s,\rho,\epsilon) \equiv k(x,s) + \rho h(x,s,\epsilon),$$
$$f(x,\rho,\epsilon) \equiv f(x) + \rho h(x,\epsilon),$$

reduces to (A_1) when $\rho = 0$ and to (A_2) when $\rho = \epsilon$. For any finite value of ρ, the Fredholm determinant and the Fredholm minors of $k(x, s, \rho, \epsilon)$ exist and are analytic for all finite values of λ. Also since $k(x, s, \rho, \epsilon)$ is analytic in ρ and possesses a resolvent for the particular value $\rho = 0$, the Fredholm determinant $D(\lambda, \rho, \epsilon)$ and the Fredholm minors

$$D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda, \rho, \epsilon \Big), \qquad (\mu = 1,2,3,\cdots),$$

are analytic for all finite values of ρ, and are not identically zero.*

Let $\lambda = \lambda_0$ be a characteristic number of multiplicity p and index q† associated with the kernel $k(x, s, 0, \epsilon) \equiv k(x, s)$. Then‡

$$(2) \qquad D(\lambda,0,\epsilon) \equiv D(\lambda) = (\lambda - \lambda_0)^p d(\lambda), \qquad d(\lambda_0) \neq 0\,;$$

and

$$(3) \qquad D\left(\begin{matrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{matrix}, \lambda, 0, \epsilon\right) \equiv D\left(\begin{matrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{matrix}, \lambda\right)$$
$$= (\lambda - \lambda_0)^{q-\mu} d_\mu\left(\begin{matrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{matrix}, \lambda\right), \ (\mu = 1,2,\cdots, q),$$

where, in general,

$$d_\mu\left(\begin{matrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{matrix}, \lambda_0\right) \neq 0.$$

* See, e.g. J. D. Tamarkin, Annals of Mathematics, (2), vol. 28 (1927), pp. 127–152.

† A characteristic number is said to be of index q if there are exactly q characteristic functions associated with it.

‡ These formulas follow as a direct consequence of the assumption that λ_0 is a characteristic number of multiplicity p and index q.

Also, for all values of μ, including $\mu=0$,*

$$\partial D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda, \rho, \epsilon\Big)/\partial\rho$$

$$= \sum_{i,j=1}^{\mu} (-1)^{i+j} h(x_i, s_j, \epsilon) D\begin{pmatrix} x_1 \cdots x_{i-1} x_{i+1} \cdots x_\mu \\ s_1 \cdots s_{j-1}\, s_{j+1} \cdots s_\mu \end{pmatrix}, \lambda, \rho, \epsilon\Big)$$

$$+ \lambda D\begin{pmatrix} x_1 \cdots x_\mu \\ s_1 \cdots s_\mu \end{pmatrix}, \lambda, \rho, \epsilon\Big) \int_a^b h(\xi, \xi, \epsilon) d\xi$$

$$(4) \quad - \lambda \sum_{i=1}^{\mu} \int_a^b h(x_i, \xi, \epsilon) D\begin{pmatrix} x_1 \cdots x_{i-1} \xi x_{i+1} \cdots x_\mu \\ s_1 \cdots s_{i-1} s_i s_{i+1} \cdots s_\mu \end{pmatrix}, \lambda, \rho, \epsilon\Big) d\xi$$

$$- \lambda \sum_{i=1}^{\mu} \int_a^b D\begin{pmatrix} x_1 \cdots x_{i-1} x_i x_{i+1} \cdots x_\mu \\ s_1 \cdots s_{i-1}\, \xi\, s_{i+1} \cdots s_\mu \end{pmatrix}, \lambda, \rho, \epsilon\Big) h(\xi, s_i, \epsilon) d\xi$$

$$- \lambda^2 \int_a^b h(\xi, \eta, \epsilon) D\begin{pmatrix} x_1 \cdots x_\mu \eta \\ s_1 \cdots s_\mu \xi \end{pmatrix}, \lambda, \rho, \epsilon\Big) d\xi d\eta.$$

Assume that there is no characteristic number of $k(x, s)$ other than $\lambda=\lambda_0$ within or on the boundary of the region $|\lambda-\lambda_0| \leqq \delta_1$. The analytic function $D(\lambda, \rho, \epsilon)$ is represented in the region $|\lambda-\lambda_0| \leqq \delta_1$; $0 \leqq \rho \leqq \delta_2 < 1$ by the convergent power series

$$(5) \quad D(\lambda, \rho, \epsilon) = \sum_{\alpha,\beta=0}^{\infty} A_{\alpha,\beta}(\epsilon)(\lambda-\lambda_0)^\alpha \rho^\beta, \qquad |A_{\alpha,\beta}(\epsilon)| \leqq A,$$

where A is a finite constant.

It follows from (2), (3), and (4) that

$$A_{\alpha,0} = 0, \ \alpha < p, \qquad A_{\alpha,\beta} = 0, \ \alpha+\beta < q, \qquad A_{p,0} \neq 0,$$

and, in general,

$$A_{\alpha,\beta} \neq 0, \quad \alpha+\beta = q.$$

We shall assume for the present that $A_{q-1,1} \neq 0$ and, by collecting terms of the same degree, express (5) in the form

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* G. C. Evans, loc. cit.

$D(\lambda,\rho,\epsilon) - D(\lambda) =$

$$[A_{q-1,1}(\lambda-\lambda_0)^{q-1}\rho + A_{q-2,2}(\lambda-\lambda_0)^{q-2}\rho^2 + \cdots + A_{0,q}\rho^q]$$
$$+ [A_{q,1}(\lambda-\lambda_0)^{q}\rho + A_{q-1,2}(\lambda-\lambda_0)^{q-1}\rho^2 + \cdots + A_{0,q+1}\rho^{q+1}]$$
$$+ \cdots .$$

Consequently, for $\lambda-\lambda_0$ and ϵ sufficiently small,*

$$|D(\lambda,\epsilon) - D(\lambda)| < 2A\epsilon[|\lambda-\lambda_0| + \epsilon]^{q-1}.$$

Hence

$$D(\lambda,\epsilon) = D(\lambda)[1 + \psi(\lambda,\epsilon)]$$

where

$$|\psi(\lambda,\epsilon)| < 2A\epsilon[|\lambda-\lambda_0| + \epsilon]^{q-1}|D(\lambda)|^{-1}.$$

Consider a circle (γ) about λ_0 of radius $r \leqq \delta_1$. Since $d(\lambda)$ in (2) can not vanish on (γ), then on (γ) $|d(\lambda)| \geqq d_0^{-1} > 0$ where d_0 is a constant different from zero. If now we choose r so that

$$2Ad_0\epsilon[r+\epsilon]^{q-1}r^{-p} = C < 1 \tag{6}$$

then on (γ), $|\psi(\lambda,\epsilon)| < 1$ and the equation $D(\lambda,\epsilon) = 0$ will have the same number of zeros within (γ) as $D(\lambda) = 0$. It is readily seen that (6) implies $r = O(\epsilon^{1/(p-q+1)})$.

THEOREM 1. *If $\lambda = \lambda_0$ is a characteristic number of (A_1) of multiplicity p and index q, then for a fixed ϵ sufficiently small (A_2) will possess exactly p characteristic numbers $\lambda = \lambda_{0,i}(\epsilon)$ $(i = 1, 2, \cdots, p)$ such that $\lambda_0 - \lambda_{0,i}(\epsilon) = O(\epsilon^{1/(p-q+1)})$ at least.*

By an analogous method it is possible to discuss the special cases that arise when $A_{q-1,1} = 0$. We shall find that if any of the coefficients $A_{\alpha,\beta}$, $\alpha+\beta = q$, are different from zero, then $r = O(\epsilon^{1/(p-q+1)})$; if all of the coefficients $A_{\alpha,\beta}, \alpha+\beta = q$, vanish, but one of the coefficients $A_{\alpha,\beta}$, $\alpha+\beta = q+1$, is different from zero, then $r = O(\epsilon^{1/(p-q)})$; if $A_{\alpha,\beta}$, $\alpha+\beta < q+2$, vanish, but one of the coefficients $A_{\alpha,\beta}$, $\alpha+\beta = q+2$ is different from zero, then $r = O(\epsilon^{1/(p+q-1)})$; and so forth.

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* $D(\lambda, \epsilon, \epsilon) \equiv D(\lambda, \epsilon)$.

THEOREM 2. *If* $\{\phi_j(x)\}$ $(j=1, 2, \cdots, q)$ *is a normalized set of characteristic functions of* (A_1) *associated with the characteristic number* $\lambda=\lambda_0$ *and* $\phi(x, \epsilon)$ *is a normalized characteristic function of* (A_2) *associated with the characteristic number* $\lambda=\lambda_{0,i}(\epsilon)$, *then there exists an approximately normalized* linear combination* $\sum_{j=1}^{q} c_j(\epsilon)\phi_j(x)$ *such that*

$$\phi(x,\epsilon) - \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x) = O(\epsilon^{1/(p-q+1)})$$

at least.

Consider the equation

$$\phi(x,\epsilon)+\lambda_{0,i}(\epsilon)\int_a^b k(x,s,\epsilon)\phi(s,\epsilon)ds = 0, \tag{7}$$

where

$$\int_a^b |\phi(x,\epsilon)|^2 dx = 1.$$

This may be written as

$$\phi(x,\epsilon) + \lambda_0\int_a^b k(x,s)\phi(s,\epsilon)ds = \Omega(x,\epsilon), \tag{8}$$

where

$$\Omega(x,\epsilon) = \lambda_0\int_a^b [k(x,s) - k(x,s,\epsilon)]\phi(s,\epsilon)ds + [\lambda_0 - \lambda_{0,i}(\epsilon)]\int_a^b k(x,s,\epsilon)\phi(s,\epsilon)ds.$$

Since

$$\lambda_{0,i}(\epsilon) - \lambda_0 = O(\epsilon^{1/(p-q+1)}), \quad k(x,s,\epsilon) - k(x,s) = O(\epsilon),$$

and since $k(x, s, \epsilon)$ is bounded and $\phi(x, \epsilon)$ is normalized, we have $\Omega(x,\epsilon) = O(\epsilon^{1/(p-q+1)})$ at least.

But (8) is a non-homogeneous equation in which λ_0 is a characteristic number and $\phi(x, \epsilon)$ is a solution. Accordingly, if $R(x, s)$ is the pseudo-resolvent of $k(x, s)$, then

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* A function $\omega(x, \epsilon)$ is said to be approximately normalized if $\int_a^b|\omega(x, \epsilon)|^2dx=1+\eta(\epsilon)$ where $\eta(\epsilon)\to 0$ as $\epsilon\to 0$.

$$\phi(x,\epsilon) = \Omega(x,\epsilon) - \lambda_0 \int_a^b R(x,s)\Omega(s,\epsilon)ds + \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x).$$

Since $R(x, s)$ is bounded,

$$(9) \quad \phi(x,\epsilon) = \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x) + \eta(x,\epsilon), \quad \eta(x,\epsilon) = O(\epsilon^{1/(p-q+1)}),$$

which yields the result

$$\int_a^b \left| \sum_{j=1}^{q} c_j(\epsilon)\phi_j(x) \right|^2 dx = 1 + \bar{\eta}(x,\epsilon), \quad \bar{\eta}(x,\epsilon) = O(\epsilon^{1/(p-q+1)}).$$

THEOREM 3. *If $u(x)$ represents a solution of (A_1) corresponding to a fixed value of the parameter which is not a characteristic number, then for ϵ sufficiently small and the same value of the parameter there is a solution $u(x, \epsilon)$ of (A_2) such that*

$$u(x,\epsilon) - u(x) = O(\epsilon).$$

If λ is not a characteristic value of (A_1), then from Theorem 1 it follows that for ϵ sufficiently small λ is not a characteristic value of (A_2). We may write (A_2) as

$$u(x,\epsilon) + \lambda \int_a^b k(x,s)u(s,\epsilon)ds = g(x,\epsilon),$$

$$(x,\epsilon) = f(x,\epsilon) - \lambda \int_a^b [k(x,s,\epsilon) - k(x,s)]u(s,\epsilon)ds.$$

Let $\mathfrak{f}(x, s, \lambda)$ represent the resolvent of $k(x, s)$. Then

$$u(x,\epsilon) + \lambda \int_a^b \mathfrak{f}(x,s,\lambda)g(s,\epsilon)ds = g(x,\epsilon),$$

or

$$u(x,\epsilon) + \lambda \int_a^b \mathfrak{f}(x,s,\lambda,\epsilon)u(s,\epsilon)ds = g(x,\lambda,\epsilon)$$

where

$$\mathfrak{f}(x,s,\lambda,\epsilon) = k(x,s,\epsilon) - k(x,s) - \lambda \int_a^b \mathfrak{f}(x,\xi,\lambda)[k(\xi,s,\epsilon) - k(\xi,s)]d\xi,$$

$$g(x,\lambda,\epsilon) = f(x,\epsilon) - \lambda \int_a^b \mathfrak{f}(x,s,\lambda)f(s,\epsilon)ds.$$

Since $\mathfrak{f}(x, s, \lambda, \epsilon) = O(\epsilon)$, the Neumann series representing the resolvent of $\mathfrak{f}(x, s, \lambda, \epsilon)$ will converge and for ϵ sufficiently small will also be $O(\epsilon)$. Consequently, $u(x, \epsilon)$ is bounded.

If we set $v(x,\epsilon) = u(x, \epsilon) - u(x)$, then from (A_1) and (A_2) it follows that

$$v(x,\epsilon) + \lambda \int_a^b k(x,s)v(s,\epsilon)ds = \eta(x,\lambda,\epsilon),$$

where

$$\eta(x,\lambda,\epsilon) = f(x,\epsilon) - f(x) - \lambda \int_a^b [k(x,s,\epsilon) - k(x,s)]u(s,\epsilon)ds = O(\epsilon).$$

Hence

$$v(x,\epsilon) = \eta(x,\lambda,\epsilon) - \lambda \int_a^b \mathfrak{f}(x,s,\lambda)\eta(s,\lambda,\epsilon)ds = O(\epsilon),$$

which proves Theorem 3.

3. *Reduction of the Differential System and the Associated Difference System to Fredholm Integral Equations.* We consider the differential system represented in matrix notation*

$$(B_1) \qquad \begin{aligned} &L(Y\cdot) + \lambda Q(x)Y\cdot = F(x)\cdot, \\ &U(Y\cdot) \equiv W_0Y(0)\cdot + W_1Y(1)\cdot = 0, \ (0 \leqq x \leqq 1), \\ &L(Y\cdot) \equiv dY\cdot/dx - A(x)Y\cdot, \end{aligned}$$

where $A(x)$, $Q(x)$, and $F(x)\cdot$ are matrices whose elements satisfy a Lipschitz condition; W_0 and W_1 are matrices of constants; and λ is a scalar parameter.

Let the interval (0, 1) be divided into m equal parts by the points $x_0 \equiv 0, x_1, x_2, \cdots, x_m \equiv 1$. Then associated with the system (B_1) is a difference system

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* Throughout the remainder of this paper capital letters—with the single exception of O—will be used to represent matrices with n rows and n columns. A dot following a capital indicates that the columns of the matrix are identical.

$$L(Y_l\cdot) + \lambda Q(x_l)Y_l\cdot = F(x_l)\cdot,$$

$$(B_2) \qquad U(Y_l\cdot) \equiv W_0 Y_0\cdot + W_1 Y_m\cdot = 0, \quad (l = 1, 2, \cdots, m),$$

$$L(Y_l\cdot) \equiv \Delta Y_{l-1}\cdot/\Delta x - A(x_l)Y_l\cdot\ ; \quad \Delta Y_l\cdot = Y_{l+1}\cdot - Y_l\cdot\ ,$$

$$\Delta x = x_{l+1} - x_l = 1/m.$$

We may assume without loss of generality that $\lambda = 0$ is not a characteristic number of either (B_1) or (B_2). Then for an arbitrary $F(x)\cdot$ there exists a unique solution of the system

$$(10) \qquad \begin{aligned} L(Y\cdot) &= F(x)\cdot, \\ U(Y\cdot) &= 0, \end{aligned} \qquad (0 \leqq x \leqq 1),$$

and this solution can be expressed in the form

$$Y(x)\cdot = \int_0^1 G(x,s)F(s)\cdot ds,$$

where $G(x, s)$ is called the Green's matrix of the system and is given explicitly by the formula*

$$G(x,s) = \overline{G}(x,s) - Y_h(x)\left[U(Y_h(x)\right]^{-1}U\left[\overline{G}(x,s)\right],$$

where

$$\overline{G}(x,s) = \begin{cases} \ \ \frac{1}{2}Y_h(x)\left[Y_h(s)\right]^{-1}, & s < x \\ -\frac{1}{2}Y_h(x)\left[Y_h(s)\right]^{-1}, & s > x \end{cases} ; \quad L(Y_h(x)) = 0.$$

Hence

$$(11) \qquad Y(x)\cdot + \lambda \int_0^1 K(x,s)Y(s)\cdot ds = \overline{F}(x)\cdot,$$

where

$$K(x,s) = G(x,s)Q(s), \qquad \overline{F}(x)\cdot = \int_0^1 G(x,s)F(s)\cdot ds,$$

is equivalent to (B_1). In the ordinary notation, the system (11) represents a system of n linear integral equations

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* Birkhoff and Langer, Proceedings American Academy of Arts and Sciences, vol. 58 (1923), pp. 51–128.

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(12) \qquad \begin{aligned} y_i(x) + \lambda \int_0^1 \sum_{j=1}^{n} k_{i,j}(x,s) y_j(s) ds &= \bar{f}_i(x), \\ k_{i,j}(x,s) = \sum_{\alpha=1}^{n} g_{i,\alpha}(x,s) q_{\alpha,j}(s), &\quad (i,j = 1,2, \cdots, n). \end{aligned}
$$

If we define

$$
k(x,s) = k_{i,j}(x - i + 1,\ s - j + 1), \quad 0 < \begin{matrix} x - i + 1 \\ s - j + 1 \end{matrix} < 1,
$$

$$
f(x) = \bar{f}_i(x - i + 1), \qquad 0 < x - i + 1 < 1,
$$

then

$$
(13) \qquad y(x) + \lambda \int_0^n k(x,s) y(s) ds = f(x),
$$

where $y(x) = y_i(x - i + 1)$, $0 < x - i + 1 < 1$, is equivalent to (12).*

It is also possible by means of a Green's matrix to reduce the system (B_2) to a single Fredholm integral equation. For this purpose we shall need the following lemmas, the proofs of which do not present any difficulty:

i. *If* $Y_h^{(m)}(x_l)$, $(l = 0, 1, 2, \cdots, m)$, *is a solution of the system*

$$
(14) \qquad L(Y_l) = 0, \qquad (l = 1,2, \cdots, m),
$$

and if† $|Y_h^{(m)}(1)| \neq 0$, *there exists a number* m_0 *such that for all* $m > m_0$, $|Y_h^{(m)}(x_l)| \neq 0$, $(l = 0, 1, 2, \cdots, m)$.

ii. *If* $|Y_h^{(m)}(1)| \neq 0$, *then for a sufficiently large* m *the most general solution of* (14) *is* $Y_h^{(m)}(x_l)C$, $(l = 0, 1, 2, \cdots, m)$, *where* C *is a matrix of constants.*

iii. *If* $\bar{Y}^{(m)}(x_l)$, $(l = 0, 1, 2, \cdots, m)$, *is a particular solution of the system*

$$
(15) \qquad L(Y_l) = F(x_l), \qquad (l = 1,2, \cdots, m),
$$

* I. Fredholm, Acta Matematica, vol. 27 (1903), pp. 378–379.

† The notation $|A|$ is used to indicate the determinant of the matrix A.

and if $|Y_h^{(m)}(1)| \neq 0$, *then for m sufficiently large*

$$Y^{(m)}(x_l) = \overline{Y}^{(m)}(x_l) + Y_h^m(x_l)C, \qquad (l = 0,1,2, \cdots, m),$$

is the most general solution of (15).

iv. *A particular solution of* (15) *is*

$$\overline{Y}^{(m)}(x_l) = \sum_{k=1}^{m} \overline{G}^{(m)}(x_l, x_{k-1})F(x_k)\Delta x, \qquad (l = 0,1,2, \cdots, m),$$

where

$$\overline{G}^{(m)}(x_l, x_{k-1}) = \begin{cases} \tfrac{1}{2}Y_h^{(m)}(x_l)[Y_h^{(m)}(x_{k-1})]^{-1}, & k \leqq l, \\ -\tfrac{1}{2}Y_h^{(m)}(x_l)[Y_h^{(m)}(x_{k-1})]^{-1}, & k > l. \end{cases}$$

By an argument closely analogous to that used by Birkhoff and Langer* for the differential system, it is readily proved that there exists a unique solution of the system

$$(16) \qquad L(Y_l\cdot) = F(x_l)\cdot, \quad U(Y_l\cdot) = 0, \quad (l = 1,2, \cdots, m),$$

and that this solution can be expressed in the form

$$Y^{(m)}(x_l)\cdot = \sum_{k=1}^{m} G^{(m)}(x_l, x_{k-1})F(x_k)\cdot\Delta x, \quad (l = 0,1,2, \cdots, m),$$

where $G^{(m)}(x_l, x_{k-1})$ is called the Green's matrix of the system and is given explicitly by the formula

$$G^{(m)}(x_l, x_{k-1}) = \overline{G}^{(m)}(x_l, x_{k-1}) - Y_h^{(m)}(x_l)\{U[Y_h^{(m)}(x_l)]\}^{-1}U[\overline{G}^{(m)}(x_l, x_{k-1})],$$
$$(l = 0,1,2, \cdots, m; \; k = 1,2, \cdots, m).$$

Consequently the system

$$(17) \quad Y^{(m)}(x_l)\cdot + \lambda \sum_{k=1}^{m} G^{(m)}(x_l, x_{k-1})Q(x_k)Y^{(m)}(x_k)\cdot\Delta x = \overline{F}^m(x_l)\cdot,$$

where

$$\overline{F}^{(m)}(x_l)\cdot = \sum_{k=1}^{m} G^{(m)}(x_l, x_{k-1})F(x_k)\cdot\Delta x,$$

is equivalent to (B_2).

* Birkhoff and Langer, loc. cit.

On setting $G^{(m)}(x_l, x_{k-1}) = G^{(m)}(x, s)$ in the intervals

$$I_x: \quad x_{l-1} < x < x_l, \qquad (l = 1, 2, \cdots, m),$$
$$I_s: \quad x_{k-1} < s < x_k, \qquad (k = 1, 2, \cdots, m),$$

and

$$\bar{F}^{(m)}(x_l)\cdot = \bar{F}^{(m)}(x)\cdot, \quad Q(x_l) = Q^{(m)}(x), \quad Y^{(m)}(x_l)\cdot = Y^m(x)\cdot,$$

in the interval I_x, we reduce (17) to an equivalent system of integral equations

$$(18) \qquad Y^{(m)}(x)\cdot + \lambda \int_0^1 K^{(m)}(x,s) Y^{(m)}(s)\cdot ds = \bar{F}^{(m)}(x)\cdot,$$
$$K^{(m)}(x,s) = G^{(m)}(x,s) Q^{(m)}(s).$$

Then (18) is equivalent to

$$(19) \qquad y^{(m)}(x) + \lambda \int_0^n k^{(m)}(x,s) y^{(m)}(s) ds = f^{(m)}(x),$$

where

$$k^{(m)}(x,s) = k_{i,j}(x - i + 1, s - j + 1), \quad 0 < \begin{matrix} x - i + 1 \\ s - j + 1 \end{matrix} < 1,$$
$$f^{(m)}(x) = \bar{f}_i^{(m)}(x - i + 1),$$
$$y^{(m)}(x) = y_i^{(m)}(x - i + 1), \qquad 0 < x - i + 1 < 1.$$

As a final step it is necessary to estimate the difference between the kernels $k(x, s)$ and $k^{(m)}(x, s)$. For this purpose we shall need the additional lemmas:

v. *If $U_1(x)$ and $U_2(x)$ are continuous matrices which, except at a finite number of points, satisfy the relations*

$$dU_1/dx = A(x) U_1 + \Omega_1, \quad U_1(1) = C_0,$$
$$dU_2/dx = A(x) U_2 + \Omega_2, \quad U_2(1) = C_0,$$

where $A(x) \ll a_0$; $\Omega_1 \ll \epsilon_1$ and $\Omega_2 \ll \epsilon_2$, ϵ_1 and ϵ_2 being positive scalars; then*

* $A << B$ indicates that each element of the matrix A is less in absolute value than the corresponding element of the matrix B. Similarly, $A << b$ indicates that each element of the matrix A is less in absolute value than the scalar b.

$$U_1(x) - U_2(x) \ll (\epsilon_1 + \epsilon_2)(e^{a_0 n x} - 1)/a_0 n.^*$$

vi. *The elements of the matrices*

$$Y_h(x), \quad [Y_h(x)]^{-1}, \quad U[Y_h(x)], \quad \{U[Y_h(x)\}^{-1}$$

are bounded; the same is true of the elements of the matrices

$$Y_h^{(m)}(x_l), \quad [Y_h^{(m)}(x_l)]^{-1}, \quad U[Y_h^{(m)}(x_l)], \quad \{U[Y_h^{(m)}(x_l)]\}^{-1},$$

$$U[\overline{G}^{(m)}(x_l, x_{k-1})], \quad (l = 0,1,2, \cdots, m\ ;\ k = 1,2, \cdots, m),$$

for m sufficiently large.†

vii. $\quad Y_h^{(m)}(x_l) - Y_h(x) = O(1/m) \quad in \quad I_x$.‡

Let the continuous matrix $Y_h^{(m)}(x)$ be defined in I_x by the relation

$$dY_h^{(m)}(x)/dx = \Delta Y_h^{(m)}(x_{l-1})/\Delta x = A(x_l)Y_l,$$

where $Y_l = Y_h^{(m)}(x_l)$. Then

$$dY_h^{(m)}(x)/dx = A(x)Y_h^{(m)}(x) + \Omega^{(m)}(x),$$

where

$$\Omega^{(m)}(x) = A(x_l)[Y_h^{(m)}(x_l) - Y_h^{(m)}(x)] + [A(x_l) - A(x)]Y_h^{(m)}(x).$$

It is evident that

$$\Omega^{(m)}(x) = O(1/m).$$

The lemma now follows as a consequence of (v).

viii. $\quad \overline{G}^{(m)}(x_l, x_{k-1}) - \overline{G}(x,s) = O(1/m) \quad in \quad I_x, I_s$.

ix. $\quad U[Y_h^{(m)}(x_l)] - U[Y_h(x)] = O(1/m)$.

x. $\quad \{U[Y_h^{(m)}(x_l)]\}^{-1} - \{U[Y_h(x)]\}^{-1} = O(1/m)$.

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* The method of proof is similar to that used in the case of a single differential equation. (Ch. J. de la Vallée Poussin, *Cours d'Analyse Infiniteésimale*, vol. 2, ed. 4, p. 135.)

† See Birkhoff and Langer, loc. cit.

‡ $A(\epsilon) = O(\epsilon)$ if each element of $A(\epsilon)$ is $O(\epsilon)$.

Combining lemmas (v) to (x) and using the fact that $Q(x)$ and $F(x)\cdot$ satisfy a Lipschitz condition, we obtain the final lemmas:

$$\text{xi.} \quad G^{(m)}(x_l, x_{k-1}) - G(x,s) = O(1/m) \quad in \quad I_x, I_s.$$

$$\text{xii.} \quad K^{(m)}(x,s) - K(x,s) = O(1/m) ;$$

$$k^{(m)}(x,s) - k(x,s) = O(1/m) ;$$

$$f^{(m)}(x) - f(x) = O(1/m).$$

Equations (13) and (19) can now be treated by methods developed in §2.

4. *Reduction of the Integro-Differential System and the Associated Difference System to Fredholm Integral Equations.* Let us consider the integro-differential system

$$(C_1) \qquad \begin{aligned} L(Y\cdot) + \lambda Q(x)Y\cdot &= F(x)\cdot + \int_0^1 H(x,s)Y(s)\cdot ds, \\ U(Y\cdot) &= 0, \end{aligned}$$

where $L(Y\cdot)$, $U(Y\cdot)$, $Q(x)$, $F(x)\cdot$, and λ have been defined above, and $H(x, s)$ satisfies a Lipschitz condition in (x, s).

Associated with this system is a difference system

$$(C_2) \qquad \begin{aligned} L(Y_l\cdot) + \lambda Q(x_l)Y_l\cdot &= F(x_l)\cdot + \sum_{k=0}^{m} H(x_l, x_k)Y_k\cdot \Delta x, \\ U(Y_l\cdot) &= 0, \qquad (l = 1, 2, \cdots, m). \end{aligned}$$

We can assume without loss of generality that $\lambda = 0$ is not a characteristic number* of either (C_1) or the differential system

* The discussion of characteristic numbers and characteristic functions of a single integro-differential equation satisfying boundary conditions, has been given by J. D. Tamarkin (Transactions of this Society, vol. 29 (1927), pp. 775–800). His results have been extended to a system of integro-differential equations by F. C. Jonah in a thesis to be submitted to Brown University in candidacy for the degree of Doctor of Philosophy.

(B_1). Since $\lambda = 0$ is not a characteristic number of (C_1), the system

$$(20)\qquad \begin{aligned} L(Y\cdot) &= F(x)\cdot + \int_0^1 H(x,s)Y(s)\cdot ds, \\ U(Y\cdot) &= 0, \end{aligned}$$

possesses a unique solution for an arbitrary $F(x)\cdot$ and this can be expressed in the form $Y(x)\cdot = \int_0^1 \Gamma(x, s)F(s)\cdot ds$. The matrix $\Gamma(x, s)$ is uniquely determined at its points of continuity and is called the Green's matrix of (20). It is readily seen that $\Gamma(x, s)$ is given explicitly by the formula

$$\Gamma(x,s) = G(x,s) + \int_0^1 \mathfrak{H}(x,\xi)G(\xi,s)d\xi,$$

if $\mathfrak{H}(x, s)$ is the resolvent of the matrix $\overline{H}(x, s)$, and

$$(21)\qquad \begin{aligned} Y(x)\cdot &= \int_0^1 \overline{H}(x,s)Y(s)\cdot ds + \overline{F}(x)\cdot, \\ \overline{H}(x,s) &= \int_0^1 G(x,\xi)H(\xi,s)d\xi, \\ \overline{F}(x)\cdot &= \int_0^1 G(x,s)F(s)\cdot ds. \end{aligned}$$

Hence

$$(22)\quad Y(x)\cdot + \lambda\int_0^1 \Gamma(x,s)Q(s)Y(s)\cdot ds = \int_0^1 \Gamma(x,s)F(s)\cdot ds$$

is equivalent to (C_1).

Similarly, an explicit expression for the Green's matrix of the system

$$(23)\qquad \begin{aligned} L(Y_l\cdot) &= F(x_l)\cdot + \sum_{k=0}^{m} H(x_l, x_k)F(x_k)\cdot\Delta x, \\ U(Y_l\cdot) &= 0, \qquad\qquad (l = 1,2,\cdots, m), \end{aligned}$$

can be obtained. Since $\lambda = 0$ is not a characteristic number of

(B_1), then from Theorem 1 it follows that for m sufficiently large $\lambda=0$ is not a characteristic number of (B_2). Hence

$$Y^{(m)}(x_l)\cdot = \sum_{k=0}^{m}\overline{H}^{(m)}(x_l,x_k)Y^{(m)}(x_k)\cdot\Delta x+\overline{F}(x_l)\cdot, \qquad (l=0,1,2,\cdots,m),$$

$$\overline{H}^{(m)}(x_l,x_k) = \sum_{j=1}^{m}G^{(m)}(x_l,x_{j-1})H(x_j,x_k)\Delta x,$$

$$\overline{F}(x_l)\cdot = \sum_{k=1}^{m}G^{(m)}(x_l,x_{k-1})F(x_k)\cdot\Delta x,$$

or

$$(24) \qquad Y^{(m)}(x)\cdot = \int_0^1 \overline{H}^{(m)}(x,s)Y^{(m)}(s)ds+\overline{F}^{(m)}(x)\cdot,$$

where

$$H^{(m)}(x,s) = H(x_l,x_k) \quad \text{in} \quad I_x, I_s,$$

$$\overline{H}^{(m)}(x,s) = \int_0^1 G^{(m)}(x,\xi)H^{(m)}(\xi,s)d\xi = \overline{H}^{(m)}(x_l,x_k) \ \ in \ \ I_x, I_s,$$

$$\overline{F}^{(m)}(x) = \int_0^1 G^{(m)}(x,s)F^{(m)}(s)\cdot ds = \overline{F}(x_l)\cdot \ \ in \ \ I_x,$$

is equivalent to (23).

In virtue of lemma (xi) and the assumption that $H(x, s)$ satisfies a Lipschitz condition, we have the lemma

xiii. $$\overline{H}^{(m)}(x,s) - H(x,s) = O(1/m),$$
$$\overline{F}^{(m)}(x)\cdot - \overline{F}(x)\cdot = O(1/m).$$

In consequence of Theorem 1, since (21) has a unique solution, then for m sufficiently large (24) will also possess a unique solution. So, for m sufficiently large, there exists a resolvent matrix $\mathfrak{H}^{(m)}(x, s)$ associated with $\overline{H}^{(m)}(x, s)$, and

$$Y^{(m)}(x)\cdot = \int_0^1 \Gamma^{(m)}(x,s)F^{(m)}(s)\cdot ds,$$

where

$$\Gamma^{(m)}(x,s) = G^{(m)}(x,s) + \int_0^1 \mathfrak{H}^{(m)}(x,\xi)G^{(m)}(\xi,s)d\xi.$$

Hence

$$(25) \qquad Y^{(m)}(x)\cdot + \lambda \int_0^1 \Gamma^{(m)}(x,s)Q^{(m)}(s)Y^{(m)}(s)\cdot ds = \int_0^1 \Gamma^{(m)}(x,s)F^{(m)}(s)\cdot ds,$$

is equivalent to (C_2), for $l<m$.

It is necessary to estimate the order of difference between the Green's matrices $\Gamma^{(m)}(x, s)$ and $\Gamma(x, s)$. For this purpose we need the lemma

xiv. $$\mathfrak{H}^{(m)}(x,s) - \mathfrak{H}(x,s) = O(1/m).$$

The resolvent matrices $\mathfrak{H}^{(m)}(x, s)$, $\mathfrak{H}(x, s)$ and the kernels $\overline{H}^{(m)}(x, s)$, $\overline{H}(x, s)$ satisfy the relations

$$\mathfrak{H}(x,s) - \overline{H}(x,s) = \int_0^1 \overline{H}(x,\xi)\mathfrak{H}(\xi,s)d\xi = \int_0^1 \mathfrak{H}(x,\xi)\overline{H}(\xi,s)d\xi,$$

$$\mathfrak{H}^{(m)}(x,s) - \overline{H}^{(m)}(x,s) = \int_0^1 \overline{H}^{(m)}(x,\xi)\mathfrak{H}^{(m)}(\xi,s)d\xi = \int_0^1 \mathfrak{H}^{(m)}(x,\xi)\overline{H}^{(m)}(\xi,s)d\xi.$$

Let

$$\overline{H}^{(m)}(x,s) - \overline{H}(x,s) = H_1^{(m)}(x,s) = O(1/m),$$
$$\mathfrak{H}^{(m)}(x,s) - \mathfrak{H}(x,s) = \mathfrak{U}^{(m)}(x,s).$$

Then

$$(26) \qquad \begin{aligned} \mathfrak{U}^{(m)}(x,s) &= \int_0^1 \overline{H}^{(m)}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi + \mathfrak{F}_1^{(m)}(x,s), \\ \mathfrak{F}_1^{(m)}(x,s) &= H_1^{(m)}(x,s) + \int_0^1 H_1^{(m)}(x,\xi)\mathfrak{H}(\xi,s)d\xi = O(1/m). \end{aligned}$$

The equation (26) is equivalent to

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 \overline{H}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi + \mathfrak{F}_2^{(m)}(x,s) ,$$

$$\mathfrak{F}_2^{(m)}(x,s) = \mathfrak{F}_1^{(m)}(x,s) + \int_0^1 H_1^{(m)}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi.$$

But $\overline{H}(x, s)$ possesses a resolvent $\mathfrak{H}(x, s)$. Hence

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 H_2^{(m)}(x,\xi)\mathfrak{U}^{(m)}(\xi,s)d\xi + \mathfrak{F}_3^{(m)}(x,s),$$

where

$$H_2^{(m)}(x,\mathrm{s}) = H_1^{(m)}(x,\mathrm{s}) + \int_0^1 \mathfrak{H}(x,\xi)H_1^{(m)}(\xi,s)d\xi = O(1/m),$$

$$\mathfrak{F}_3^{(r)}(x,s) = \mathfrak{F}_1^{(m)}(x,s) + \int_0^1 \mathfrak{H}(x,\xi)\mathfrak{F}_1^{(m)}(\xi,s)d\xi = O(1/m).$$

But for m sufficiently large, the elements of $\mathfrak{H}_2^{(m)}(x, \xi)$, the resolvent matrix of $H_2^{(m)}(x, \xi)$, are bounded, and

$$\mathfrak{U}^{(m)}(x,s) = \int_0^1 \mathfrak{H}_2^{(m)}(x, \xi)\mathfrak{F}_3^{(m)}(\xi,s)d\xi + \mathfrak{F}_3^{(m)}(x,s),$$

whence

$$\mathfrak{U}^{(m)}(x,s) = O(1/m).$$

In virtue of this lemma,

$$\Gamma^{(m)}(x,s) - \Gamma(x,s) = O(1/m).$$

We have shown that the systems (C_1) and (C_2) can be reduced respectively to the integral equations (22) and (25), respectively which possess the same character as the integral equations (11) and (18). So, (22) and (25) can be reduced to Fredholm integral equations to which the theorems of §2 apply.

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