Assume that $S-R_1'$ contains a point, say U, of M_2. Then by Condition 2 there exists an arc UE_3 contained in M_2+E_3. This is impossible for such an arc must contain a point of J_1. Therefore M_2 is contained in R_1. Similarly it can be proved that M_2 is contained in $S-R_2'$ and so in R_3. Similarly it can be proved that M_2 is contained in $S-R_3'$. Thus a contradiction is obtained. Therefore there do not exist three points of L.

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TWO CLASSES OF PERIODIC ORBITS WITH REPELLING FORCES*

BY T. H. RAWLES

1. *Introduction.* The purpose of this paper is to obtain certain classes of periodic orbits† of a system consisting of one body of very great mass which attracts two mutually repellent bodies of very small mass. The forces of attraction and repulsion in the system are assumed to vary inversely as the squares of the distances. It will be shown that if certain conditions of symmetry are imposed the orbits of the two small bodies must lie either in two parallel planes or else be coplanar. In the former case they are circles about a line perpendicular to the two planes and passing through the large body. For the second type of orbit a development in powers of a parameter will be obtained.

2. *The Equation of Motion.* By a proper choice of the units of time and space we can make the coefficient of the intensity of the attraction equal to 1. Then if K^2 is the

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ratio of the repulsion to the attraction the force function of the system is

$$(1) \qquad U = \frac{1}{r_1} + \frac{1}{r_2} - \frac{K^2}{\Delta},$$

where r_1 and r_2 represent the distances between the two small bodies and the large body while Δ is the distance between the small bodies.

The unit of mass may be so chosen that the mass of each of the small bodies is unity. Then let m denote the mass of the large body. The system will be referred to the rectangular axes ξ, η, ζ, and we shall indicate the coordinates of the large body by the subscript 0, and the coordinates of the small bodies by the subscripts 1 and 2. With this choice of units and coordinates, the equations of motion are

$$(2) \qquad m\ddot{\xi}_0 = \frac{\partial U}{\partial \xi_0}, \quad m\ddot{\eta}_0 = \frac{\partial U}{\partial \eta_0}, \quad m\ddot{\zeta}_0 = \frac{\partial U}{\partial \zeta_0};$$

$$(3) \qquad \ddot{\xi}_i = \frac{\partial U}{\partial \xi_i}, \quad \ddot{\eta}_i = \frac{\partial U}{\partial \eta_i}, \quad \ddot{\zeta}_i = \frac{\partial U}{\partial \zeta_i}, \quad (i = 1, 2).$$

If m is very large, the effect of the attraction of the other bodies on the large body is very small. We shall assume that this can be neglected altogether. In other words we assume that the mass of the large body is infinite as compared to that of the other two. Under this assumption we can neglect the right members of (2). These equations then show that the large body moves through space with uniform velocity in a straight line.

We now take this body as the origin, and denote the coordinates of the other two by x_i, y_i, z_i, $(i = 1, 2.)$ The equations of motion of the two small bodies then become

$$(4) \qquad \ddot{x}_i = \frac{\partial U}{\partial x_i}, \quad \ddot{y}_i = \frac{\partial U}{\partial y_i}, \quad \ddot{z}_i = \frac{\partial U}{\partial z_i},$$

where

$$(5) \qquad U = \frac{1}{r_1} + \frac{1}{r_2} - \frac{K^2}{\Delta},$$

and

$$(6) \quad \begin{cases} r_i^2 = x_i^2 + y_i^2 + z_i^2, & (i = 1,2), \\ \Delta^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2. \end{cases}$$

3. *Periodic Orbits.* The periodic orbits with which we are concerned are those for which

$$(7) \quad y_1 - y_2 = 0, \quad z_1 - z_2 = 0, \quad x_1 - x_2 = 0.$$

With these restrictions equations (4) now become

$$(8) \quad \begin{cases} \ddot{x}_1 = \dfrac{-x_1}{r_1^3} + \dfrac{k^2(x_1 - x_2)}{((x_1 - x_2)^2)^{3/2}} = \dfrac{-x_1}{r_1^3} + \dfrac{k^2}{(2x_1)^2}, \\ \ddot{y}_1 = \dfrac{-y_1}{r_1^3}, \quad \ddot{z}_1 = \dfrac{-z_1}{r_1^3}, \end{cases}$$

with three similar equations for the coordinates of the other particle. The variables are now separated and we need only consider equations (8). When these are solved the motion of the other particle can be obtained from (7).

Let us now put $x_1 = x$, $y_1 = y$, $z_1 = z$, $r_1 = r$, and $k^2/4 = 1 - e^2$. Our equations may then be written in the form

$$(9) \quad \ddot{x} = \dfrac{-x}{r^3} + \dfrac{1 - e^2}{x^2}, \quad \ddot{y} = \dfrac{-y}{r^3}, \quad \ddot{z} = \dfrac{-z}{r^3}.$$

Two integrals of (9) can be found at once. They are

$$(10) \quad \dfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \dfrac{1}{r} - \dfrac{1 - e^2}{x} + C, \quad \dot{z}y - z\dot{y} = h.$$

Also we notice that if a solution

$$x = X(e,t), \quad y = Y(e,t), \quad z = Z(e,t)$$

is known, then another solution is

$$x = aX\left(e, \dfrac{t - t_0}{a^{3/2}}\right), \quad y = aY\left(e, \dfrac{t - t_0}{a^{3/2}}\right), \quad z = aZ\left(e, \dfrac{t - t_0}{a^{3/2}}\right).$$

4. *Periodic Solutions of Period* 2π.   We shall obtain
solutions periodic with the period 2π.   From the remark
which we have just made we can infer the existence of
solutions of any desired period. The existence of a solution
developed in powers of e as a parameter follows from a
theorem of MacMillan.* When $e = 0$, a solution of the equa-
tions is

(11) $x = 1,\quad y = 0,\quad z = 0.$

The solutions which we shall proceed to develop are of the
form

$$\begin{cases} x = 1 + e^2 x_2 + e^4 x_4 + \cdots, \\ y = e y_1 + e^3 y_3 + \cdots, \\ z = e z_1 + e^3 z_3 + \cdots. \end{cases}$$

When developed in powers of the parameter, the equations
of motion are

(12) $\begin{cases} \sum_0 e^{2i} \ddot{x}_{2i} = [\tfrac{3}{2}(y_1^2 + z_1^2) - 1]e^2 + \cdots, \\[2mm] \sum_0 e^{2i+1} \ddot{y}_{2i+1} = -y_1 e + [-y_3 \\[2mm] \qquad\qquad\qquad + \tfrac{3}{2} y_1(2x_2 + y_1^2 + z_1^2)]e^3 + \cdots, \\[2mm] \sum_0 e^{2i+1} \ddot{z}_{2i+1} = -z_1 e + [-z_3 \\[2mm] \qquad\qquad\qquad + \tfrac{3}{2} z_1(2x_2 + y_1^2 + z_1^2)]e^3 + \cdots. \end{cases}$

Equating the coefficients of e in the last two equations we
obtain

(13) $\ddot{y}_1 + y_1 = 0,\quad \ddot{z}_1 + z_1 = 0.$

The solutions of these equations are

(14) $\begin{cases} y_1 = c_1 \cos t + c_2 \sin t, \\ z_1 = c_3 \cos t + c_4 \sin t, \end{cases}$

where $c_1, \cdots, c_4$ are constants of integration.   From the
coefficients of e in the first of (12), we have

* Transactions of this Society, vol. 13 (1912), p. 157.

$$(15) \begin{cases} \ddot{x}_2 = \dfrac{3}{2}(y_1{}^2 + z_1{}^2) - 1 \\[2mm] \quad = \dfrac{3}{2}\left[\left(\dfrac{c_1{}^2 + c_2{}^2 + c_3{}^2 + c_4{}^2}{2}\right)\right. \\[2mm] \qquad \left. + \left(\dfrac{c_1{}^2 + c_3{}^2 - c_2{}^2 - c_4{}^2}{2}\right)\cos 2t \right. \\[2mm] \qquad \left. + (c_1 c_2 + c_3 c_4)\sin 2t\right] - 1. \end{cases}$$

Upon integration we obtain

$$x_2 = \left(\frac{3}{4}(c_1{}^2 + c_2{}^2 + c_3{}^2 + c_4{}^2) - 1\right)\frac{t^2}{2} + c_5 t + c_6$$

$$\qquad - \frac{3}{16}(c_1{}^2 + c_3{}^2 - c_2{}^2 - c_4{}^2)\cos 2t - \frac{3}{8}(c_1 c_2 + c_3 c_4)\sin 2t,$$

c_5 and c_6 being arbitrary.

The periodicity conditions which we shall impose are that $x(2\pi) - x(0) = 0$ and $\dot{x}(2\pi) - \dot{x}(0) = 0$, with similar conditions for the other coordinates. Moreover, the coefficient of each power of the parameter must satisfy these conditions. When the periodicity conditions are applied to x_2 and $\dot{x}_2$ we see that

$$\left(\frac{3}{4}(c_1{}^2 + c_2{}^2 + c_3{}^2 + c_4{}^2) - 1\right)\frac{(2\pi)^2}{2} - 2\pi c_5 = 0,$$

$$\left(\frac{3}{4}(c_1{}^2 + c_2{}^2 + c_3{}^2 + c_4{}^2) - 1\right)(2\pi) = 0.$$

From these relations we now find that

$$(16) \qquad c_1{}^2 + c_2{}^2 + c_3{}^2 + c_4{}^2 = 4/3, \quad c_5 = 0.$$

Obviously the periodicity conditions imposed on y_1, $\dot{y}_1$, z_1 and $\dot{z}_1$ are satisfied identically.

When we proceed to the next approximation we have the equations

(17)
$$\begin{cases} \ddot{y}_3 + y_3 = (3/2)(2x_2 + y_1{}^2 + z_1{}^2)y_1, \\ \ddot{z}_3 + z_3 = (3/2)(2x_2 + y_1{}^2 + z_1{}^2)z_1. \end{cases}$$

These equations become

$$\ddot{y}_3 + y_3 = \sum_i A_i{}^{(3)} \sin it + \sum B_i{}^{(3)} \cos it,$$

$$\ddot{z}_3 + z_3 = \sum_i C_i{}^{(3)} \sin it + \sum D_i{}^{(3)} \cos it, \quad (i = 1,3),$$

where the coefficients of the periodic terms represent known functions of the constants of integration. Upon integrating the first of these two equations, we have

$$y_3 = c_1{}^{(3)} \cos t + c_2{}^{(3)} \sin t - A_1{}^{(3)} \frac{t \cos t}{2} + B_1{}^{(3)} \frac{t \sin t}{2}$$

$$+ \text{ terms in } \cos 3t \text{ and } \sin 3t,$$

where $c_1{}^{(3)}$ and $c_2{}^{(3)}$ are the new constants of integration. When we impose the condition that y_3 and $\dot{y}_3$ shall be periodic we find that $A_1{}^{(3)} = B_1{}^{(3)} = 0$. In the same manner we can show that $C_1{}^{(3)} = D_1{}^{(3)} = 0$.

5. *Restrictions on the Constants.* These four equations impose new restrictions on the constants c_i. They become

(18)
$$\begin{cases} B_1{}^{(3)} = 3(c_6 + \tfrac{1}{3})c_1 + c_1 M + c_2 N = 0, \\ A_1{}^{(3)} = 3(c_6 + \tfrac{1}{3})c_2 - c_2 M + c_1 N = 0, \\ D_1{}^{(3)} = 3(c_6 + \tfrac{1}{3})c_3 + c_3 M + c_4 N = 0, \\ C_1{}^{(3)} = 3(c_6 + \tfrac{1}{3})c_4 - c_4 M + c_3 N = 0, \end{cases}$$

where

$$M = \tfrac{3}{32}(c_1{}^2 + c_3{}^2 - c_2{}^2 - c_4{}^2), \quad N = \tfrac{3}{16}(c_1 c_2 + c_3 c_4).$$

When we multiply the four equations by c_4, $-c_3$, $-c_2$, c_1, respectively and add, the result is

(19)
$$3(c_6 + \tfrac{1}{3})(c_1 c_4 - c_2 c_3) = 0.$$

There are three cases according as one factor is zero or both factors are zero.

CASE I. $c_6 + 1/3 = 0$, $c_1 c_4 - c_2 c_3 \neq 0$.
The first of (16) shows that not all of the constants can be
zero. It then follows from equations (18) that $M = N = 0$.
This gives us the two equations

$$c_1^2 + c_3^2 - c_2^2 - c_4^2 = 0, \quad c_1 c_2 + c_3 c_4 = 0.$$

If, for example, $c_2 \neq 0$ we have from the second of these two
equations

$$c_1^2 = \frac{c_3^2 c_4^2}{c_2^2}.$$

When this is substituted in the first it becomes

$$c_3^2 c_4^2 + c_2^2 c_3^2 - c_2^4 - c_4^2 c_2^2 = 0,$$

or

$$(c_2^2 + c_4^2)(c_3^2 - c_2^2) = 0.$$

Since $c_2 \neq 0$ we must have

$$c_3 = c_2, \quad c_1 = -c_4, \quad \text{or} \quad c_1 = c_4, \quad c_3 = -c_2.$$

Case I shows that as far as the approximation has been
carried the motion is in a circle in a plane normal to the
x axis. In fact we have

$$y_1^2 + z_1^2 = c_1^2 + c_2^2 = \tfrac{2}{3},$$

$$x_2 = -\tfrac{1}{3}.$$

CASE II. $c_6 + 1/3 = 0$, $c_2 c_3 - c_1 c_4 = 0$.
It is easy to show that this case is impossible. We would
have as in Case I $M = N = 0$, and would obtain the same
relations between the constants. When these relations are
substituted in $c_2 c_3 - c_1 c_4 = 0$ that equation becomes $c_2^2 + c_1^2 = 0$,
which is impossible since not all of these constants can be
zero.

CASE III. $c_6 + 1/3 \neq 0$, $c_2 c_3 - c_1 c_4 = 0$.
On multiplying the first of equations (14) by c_3, the second
by c_1 and subtracting, since $c_2 c_3 - c_1 c_4 = 0$, we obtain $c_3 y_1$
$- c_1 z_1 = 0$, which shows that the motion lies in a plane passing
through the x axis. (See Fig. 1.)

6. *The Orbits.* We shall next show that when the process is continued each approximation gives rise to terms of the same type as those already found, and, therefore, that the periodic orbits are either circles about the x axis, or else they lie entirely in a plane containing the x axis.
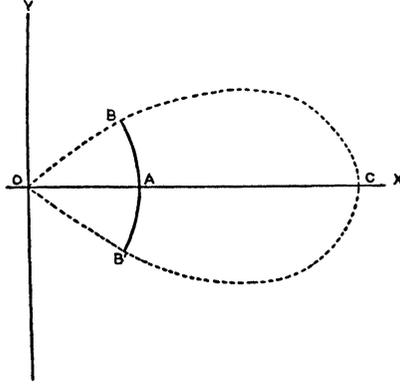

FIG. 1

The figure illustrates the path described by one of the particles for the case in which $e = 1/2$. When $t = 0$ it is at A. It reaches B when $t = \pi/2$, stops, and then returns along the same path to A, $t = \pi$. It then continues to B', $t = 3\pi/2$, and finally completes the cycle by returning to A when $t = 2\pi$. The position of the other particle is at all times symmetric to that of this one with respect to the y axis. The curve of zero velocity (conchoid) is indicated by $OBCB'O$.

For the discussion of Case I, let us suppose that when $t = 0$, $y = 0$. We then have

$$c_2 = c_3 = (\tfrac{2}{3})^{1/2}, \quad c_4 = -c_1 = 0.$$

The solution, as far as we have carried it, is

$$y = e(c_2 \sin t) + e^3(c_2^{(3)} \sin t),$$
$$z = e(c_2 \cos t) + e^3(c_3^{(3)} \cos t + c_4^{(3)} \sin t),$$
$$x = 1 + e^2(-\tfrac{1}{3}).$$

The equation for x_4 becomes

$$\ddot{x}_4 = 3(y_1 y_3 + z_1 z_3) + 2x_2 + \tfrac{9}{2} x_2^2$$
$$= (\tfrac{3}{2}(c_2 c_2^{(3)} + c_2 c_3^{(3)}) - \tfrac{1}{6})$$
$$+ \tfrac{3}{2} \{ c_2(c_3^{(3)} - c_2^{(3)}) \} \cos 2t + \tfrac{3}{2} c_2 c_4^{(3)} \sin 2t.$$

Periodicity requires that

$$c_2(c_2^{(3)} + c_3^{(3)}) = 1/9,$$

and we have upon integration

$$x_4 = c_6^{(4)} - \tfrac{3}{8} c_2(c_3^{(3)} - c_2^{(3)}) \cos 2t - \tfrac{3}{8} 8 c_2 c_4^{(3)} \sin 2t.$$

The equations for y_5 and z_5 now become

$$\ddot{y}_5 + y_5 = \tfrac{3}{2} y_1(2x_4 + x_2^2 + 2(y_1 y_3 + z_1 z_3)),$$
$$\ddot{z}_5 + z_5 = \tfrac{3}{2} z_1(2x_4 + x_2^2 + 2(y_1 y_3 + z_1 z_3)),$$

which are when written out in full

$$\ddot{y}_5 + y_5 = \tfrac{3}{2}(c_2 \sin t) \{ (\tfrac{2}{9} + 2c_6^{(4)})$$
$$+ \tfrac{1}{4} c_2(c_3^{(3)} - c_2^{(3)}) \cos 2t + \tfrac{1}{4} c_2 c_4^{(3)} \sin 2t \},$$
$$\ddot{z}_5 + z_5 = \tfrac{3}{2}(c_2 \cos t) \{ (\tfrac{2}{9} + 2c_6^{(4)})$$
$$+ \tfrac{1}{4} c_2(c_3^{(3)} - c_2^{(3)}) \cos 2t + \tfrac{1}{4} c_2 c_4^{(3)} \sin 2t \}.$$

When the products of the· trigonometric functions are expressed as functions of multiple angles it will be seen that for periodicity we must have

$$c_6^{(4)} = -1/9, \quad c_3^{(3)} = c_2^{(3)}, \quad c_4^{(3)} = 0.$$

The last of these conditions shows that z_3 contains no term in $\sin t$. The complementary function for z_5 may contain a term in $\sin t$ but the next integration will show that the coefficient of this term is zero.

The process can be continued indefinitely and our solution is of the form

$$y = p_1 \sin t, \quad z = p_1 \cos t, \quad x = p_2,$$

where p_1 and p_2 are power series in e. The orbit is obviously a circle about the x axis.

We could have obtained this orbit by more elementary means. If we put

$$x = (1 - e^2)^{1/3},$$

$$y = (1 - (1 - e^2)^{2/3})^{1/2} \sin (t - t_0),$$

$$z = (1 - (1 - e^2)^{2/3})^{1/2} \cos (t - t_0),$$

it is easily seen that the equations of motion are satisfied.

For the discussion of the case where the motion, as far as the first approximation is concerned, lies in a plane containing the x axis, let us choose that plane $z = 0$. We shall take as the origin of time the instant when $y = 0$. With this choice we have $c_2 = (4/3)^{1/2}$, $c_1 = c_3 = c_4 = 0$.

Under these restrictions the equation for z_3 is $\ddot{z}_3 + z_3 = 0$, and its solution is

$$z_3 = c_3^{(3)} \cos t + c_4^{(3)} \sin t.$$

The constants of integration introduced in the complementary function of x_2 have already been determined by (16) and (18), and the equation for z_5 now becomes

$$\ddot{z}_5 + z_5 = z_3(3x_2 + \tfrac{3}{2}y_1^2)$$

$$= - \tfrac{1}{4}c_3^{(3)} \cos t - \tfrac{1}{8}c_3^{(3)} \cos 3t - \tfrac{1}{8}c_4^{(3)} \sin 3t.$$

Now for a periodic solution it is obvious that $c_3^{(3)} = 0$. Therefore z_3 contains only a term in $\sin t$. If we continue this process imposing each time the condition that when $t = 0$, $y = 0$, we shall find that z as well as y contains only terms in $\sin (2k+1)t$. This means of course that the particle crosses the x axis when $t = n\pi$. The motion will always remain in the plane which contains the x axis and the tangent to the orbit at the instant the particle crosses that axis because all the forces are acting in this plane. This can be easily verified by a consideration of the differential equations.

7. *Indefinite Extension of the Process.* It remains to show that the process of determining the constants can be carried on indefinitely. For this purpose we may take $z = 0$ as the

plane of motion and suppose that the following terms have been calculated:

$$
(20) \quad
\begin{cases}
x_{2j} = \sum_0^j a_{2i}^{(2j)} \cos 2it, & (j = 0, 1, 2, \cdots, n), \\[2ex]
y_{2j+1} = d_{2j+1} \sin t + \sum_1^j a_{2i+1}^{(2j+1)} \sin (2i + 1)t.
\end{cases}
$$

All the coefficients have been determined except d_{2n+1}. When we equate the coefficients of e^{2n+2} in the equation for x we find

$$
\ddot{x}_{2n+2} = x_{2n+2}(-1 + 3 - 2) + 3y_1 y_{2n+1} + \text{known terms}.
$$

This may be written in the form

$$
(21) \qquad \ddot{x}_{2n+2} = \tfrac{3}{2} c_2 d_{2n+1} (1 - \cos 2t) + \sum_0^{n+1} A_{2i}^{(2n+2)} \cos 2it,
$$

where all the coefficients are known except d_{2n+1}. The arguments of the periodic functions must be even multiples of the time for they all enter from x or y^2 which contain only terms with arguments of that form.

The integration of this equation will produce a secular term unless

$$
(i) \qquad\qquad \tfrac{3}{2} c_2 d_{2n+1} + A_0^{(2n+2)} = 0.
$$

Having imposed this condition, we find that

$$
(22) \qquad x_{2n+2} = b_{2n+2}t + d_{2n+2} + \sum_1^{n+1} a_{2i}^{(2n+2)} \cos 2it,
$$

where b_{2n+2} and d_{2n+2} are constants of integration and

$$
a_{2i}^{(2n+2)} = \frac{- A_{2i}^{(2n+2)}}{(2i)^2} .
$$

For periodicity we must of course put

$$
(ii) \qquad\qquad\qquad b_{2n+2} = 0.
$$

We now determine y_{2n+3} by equating the coefficients of e^{2n+3} in the y equation. This gives

$$\ddot{y}_{2n+3} + y_{2n+3} = 3y_1 x_{2n+2} + \text{known terms},$$

which may be written in the form

$$(23) \quad \ddot{y}_{2n+3} + y_{2n+3} = 3c_2 d_{2n+2} \sin t + \sum_0^{n+1} A_{2i+1}^{(2n+3)} \sin (2i + 1)t.$$

The arguments of the periodic functions are odd multiples of the time because they arise from terms such as y^{2k+1} and $y^{2k+1} x^{2h}$. If the coefficient of $\sin t$ in the right member is not zero a secular term will be introduced. This requires that

$$(iii) \qquad\qquad 3c_2 d_{2n+2} + A_1^{2n+3} = 0.$$

When the equation is integrated we have

$$(24) \quad y_{2n+3} = d_{2n+3} \sin t + b_{2n+3} \cos t + \sum_1^{n+1} a_{2i+1}^{(2n+3)} \sin (2i + 1)t,$$

where

$$a_{2i+1}^{(2n+3)} = \frac{A_{2i+1}^{(2n+3)}}{1 - (2i + 1)^2}.$$

We may suppose that the origin of time has been fixed by the condition $y = 0$ when $t = 0$. This determines the remaining constant and we have

$$(iv) \qquad\qquad b_{2n+3} = 0.$$

We now have determined all the coefficients to and including x_{2n+2} and y_{2n+2}, except the coefficient of $\sin t$ in the latter. Equations (22) and (24) are of the same form as (20), with n replaced by $n+1$ and it is evident that the process can be carried on indefinitely.

8. *Development of x and y in Terms of the Parameter.* The development of x and y has been carried as far as terms involving the fifth power of the parameter. The expressions found are

$$
\begin{cases}
x = 1 + \left(-\dfrac{3}{8} + \dfrac{1}{4}\cos 2t \right) e^2 \\[2mm]
\qquad + \left(-\dfrac{391}{4608} + \dfrac{29}{384}\cos 2t - \dfrac{5}{1536}\cos 4t \right) e^4, \\[4mm]
y = (3)^{1/2} \left[\left(\dfrac{2}{3}\sin t \right) e + \left(\dfrac{1}{96}\sin 3t \right) e^3 \right. \\[2mm]
\qquad + \left(-\dfrac{3307}{331776}\sin t + \dfrac{91}{36864}\sin 3t \right. \\[2mm]
\qquad \left. \left. + \dfrac{37}{110592}\sin 5t \right) e^5 \right].
\end{cases}
\tag{25}
$$

For the case of the helium atom $k^2 = 1/2$ and the parameter e must be given the value $(7/8)^{1/2}$. For this value of the parameter the convergence is not rapid, and the terms which we have calculated do not give the same degree of accuracy as the numerical integration of Langmuir. However the two results are in sufficient agreement to verify each other.

The numerical results are easily checked by means of the energy integral

$$
\frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \frac{1}{r} - \frac{1 - e^2}{x} + C.
\tag{10}
$$

The constant of integration is itself a power series in the parameter. As far as the sixth power of the parameter, it is

$$
C = -\frac{1}{3}e^2 + \frac{1}{16}e^4 + \frac{127}{10368}e^6.
\tag{26}
$$

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