NOTE ON THE DERIVATIVE CIRCULAR CONGRUENCE OF A POLYGENIC FUNCTION*

BY EDWARD KASNER

In the first paper† I published on polygenic functions I stated the following theorem.

If

$$w = \phi(x,y) + i\psi(x,y)$$

is a polygenic function of

$$z = x + iy$$

then the first derivative of w with regard to z, $dw/dz=\gamma =\alpha+i\beta$, is represented in the γ-plane by the congruence of circles

$$(\alpha - H)^2 + (\beta - K)^2 = R^2 \tag{1}$$

where

$$\left\{\begin{aligned} 2H &= \phi_x + \psi_y, \quad 2K = - \phi_y + \psi_x, \\ R^2 &= h^2 + k^2, \\ 2h &= \phi_x - \psi_y, \quad 2k = \phi_y + \psi_x. \end{aligned}\right. \tag{2}$$

To every point (x, y) of the z-plane corresponds by means of dw/dz that circle of (1) determined by the particular pair of values (x, y).

In this paper I wish to study the converse problem. Let $H(x, y)$, $K(x, y)$, and $R(x, y)$ be three arbitrary functions of (x, y)‡ and for every pair of values (x, y) construct in the γ-plane the circle whose center is (H, K)

* Presented to the Society, September 6, 1928.

† *A new theory of polygenic functions*, Science, vol. 66 (Dec., 1927), pp. 581–582. See also a previous paper by Hedrick, Ingold, and Westfall, Journal de Mathématiques, *Theory of non-analytic functions of a complex variable*, (6), vol. 2 (1923), pp. 327–342.

‡ We require, however, that these functions be continuous and have continuous first, second, and third derivatives.

and whose radius is R. What are the necessary and sufficient conditions to be fulfilled by H, K, and R, in order that the congruence of circles so obtained may map the derivative of a polygenic function? In other words, when can we find two functions $\phi(x, y)$ and $\psi(x, y)$, such that the relations (2) expressing the correspondence between the circles of the congruence and the points of the z-plane are fulfilled?

The required conditions are found by eliminating ϕ and ψ from the three equations

$$(3) \quad \left\{ \begin{array}{l} 2H = \phi_x + \psi_y, \qquad 2K = -\phi_y + \psi_x, \\ 4R^2 = (\phi_x - \psi_y)^2 + (\phi_y + \psi_x)^2. \end{array} \right.$$

This is accomplished by forming all the first derivatives, then all the second derivatives, then all the third derivatives etc., of system (3), till at one step the total number of equations obtained is larger than the number of derivatives $\phi_x, \phi_y, \psi_x, \cdots$ occurring in them. The necessary and sufficient conditions on H, K, R are then furnished by the eliminants with regard to $\phi_x, \phi_y, \psi_x, \cdots$ of the entire system of equations.

Calculation shows that the process of differentiation has to be carried up to the third derivatives of (3); the total number of equations obtained is then thirty, while the number of derivatives $\phi_x, \phi_y, \psi_x, \cdots, \cdots \psi_{yyyy}$ is only twenty-eight. The system will therefore have two eliminants, that is, H, K, *and* R *have to obey two conditions in order that their congruence may represent the derivative of a polygenic function.*

We will not carry out the elimination directly from (3) but from another system of three independent functions of H, K and R, namely,

$$M = \tfrac{1}{2}[H_x - K_y + i(H_y + K_x)] = \tfrac{1}{2}[h_x + ik_x - i(h_y + ik_y)],$$

$$N = \tfrac{1}{2}[H_x - K_y - i(H_y + K_x)] = \tfrac{1}{2}[h_x - ik_x + i(h_y - ik_y)],$$

$$O = R^2 = h^2 + k^2.$$

We simplify this system by the following changes.

1. We use

$$u = x + iy, \quad v = x - iy^*$$

as independent variables instead of x and y; in order to be able to substitute

$$x = \frac{u + v}{2}, \quad y = \frac{u - v}{2i}$$

into ϕ and ψ, we assume these functions to be analytic, that is, developable into power series, in x and y in the region considered.†

2. We use

$$h + ik = a, \quad h - ik = b$$

instead of ϕ and ψ as the functions to be eliminated. Then O, M, N assume the simple forms

$$O = a \cdot b, \qquad M = a_u, \qquad N = b_v.$$

To carry through the elimination of a and b we start out from the following expressions:

$$(4) \qquad \begin{cases} M = a_u, & O = ab, \\ N = b_v, & O_{uv} = a_{uv}b + ab_{uv} + a_u b_v + a_v b_u, \\ O_u = a_u b + ab_u, & M_v = a_{uv}, \\ O_v = a_v b + ab_v, & N_u = b_{uv}. \end{cases}$$

From these eight equations we eliminate the six first and second derivatives of a and b which occur in them. The resulting equations are

$$(5) \qquad ab - O = 0, \quad Aa + Bb + C = 0,$$

where

$$(6) \qquad \begin{cases} A = ON_u - O_u N, \\ B = OM_v - O_v M, \\ C = O_u O_v + O(2MN - O_{uv}). \end{cases}$$

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* We use u and v instead of z and $\bar{z}$, merely to simplify the writing when these letters occur as subscripts.

† In my previous papers, ϕ and ψ were only assumed to be continuous differentiable functions of x and y.

Differentiating the second equation of (5) with regard to u and with regard to v, and replacing the first derivatives of a and b in the two expressions obtained by their values in M, N, O_u, O_v, a, b according to (4), we find

$$(7)\qquad \begin{cases} A_u a^2 + (AM + C_u)a - BMb + (BO)_u = 0, \\ B_v b^2 + (BN + C_v)b - ANa + (AO)_v = 0. \end{cases}$$

The elimination of a and b from (5) and (7) furnishes two equations in O and its derivatives up to the third order, and in M and N and their derivatives up to the second order. If in these equations we return from the complex quantities u, v, M, N to the original real quantities x, y, H, K, and from O to R, we obtain two equations containing R and the derivatives of H, K, and R up to the third order. According to what was said above, they represent the necessary and sufficient conditions that must be fulfilled by H, K, and R.

These two equations will be either mutually conjugate or each self-conjugate. This follows from the fact that u and v, a and b, M and N, A and B, are mutually conjugate, while O and C are self-conjugate, so that equations (5) are self-conjugate, while equations (7) are mutually conjugate.

The elimination of a and b from (5) and (7) can be carried through as follows. By multiplying the linear equation of (5) first by $A_u \cdot a/A$, then by $B_v \cdot b/B$, and subtracting it first from the first, then from the second, equation of (7), we reduce (5) and (7) to the simpler system

$$(8)\qquad \begin{cases} ab - O = 0, & Aa + Bb + C = 0, \\ A_1 a + B_1 b + C_1 = 0, & A_2 a + B_2 b + C_2 = 0, \end{cases}$$

where

$$A_1 = A(AM + C_u) - A_u C, \quad A_2 = -ABN,$$

$$B_1 = -ABM, \quad B_2 = B(BN + C_v) - B_v C,$$

$$C_1 = A(BO)_u - BA_u O, \quad C_2 = B(AO)_v - AB_v O.$$

Then

$$(9)\qquad \begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = 0, \quad \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \cdot \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}^2 \cdot O$$

represent the two eliminants of the system (8).

The two necessary and sufficient conditions to be fulfilled by three arbitrary functions $H(x, y)$, $K(x, y)$, *and* $R(x, y)$ *of* x *and* y *in order that the congruence of circles*

$$(\alpha - H)^2 + (\beta - K)^2 = R^2$$

which they determine in the (α, β)*-plane may represent the derivative of a polygenic function are obtained by retransforming the two equations* (9) *from the quantities* u, v, M, N, O *into the quantities* x, y, H, K, R; *the two final conditions contain* R *and the derivatives of* H, K *and* R *up to the third order.*

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ON THE INVERSION OF ANALYTIC TRANSFORMATIONS*

BY B. O. KOOPMAN†

We wish to consider the transformation

$$(1)\qquad x_i = f_i(y_1, \cdots, y_n), \qquad (i = 1, \cdots, n),$$

in the neighborhood of the origin $(y) = (0)$, at which point the functions f_i are analytic, and vanish simultaneously. We are interested in the case in which the jacobian

$$J = \frac{\partial(f_1, \cdots, f_n)}{\partial(y_1, \cdots, y_n)}$$

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* Presented to the Society under the title *Some theorems on inversion*, October 29, 1927.

The methods and point of view of the second chapter of W. F. Osgood's *Lehrbuch der Funktionentheorie* are assumed throughout this paper. Our results may be regarded as the extension of §20 of that work.

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