THE NODES OF THE RATIONAL PLANE QUARTIC*

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The projective properties of the rational quartic curve may be studied from two points of view. If the curve is given parametrically, its invariants are expressible in terms of the coefficients of the fundamental involution. If the curve is given by the plane $(KX)=0$, and the Steiner quartic surface $(X^{1/2})=0$, the invariants are expressible in symmetric functions of the coefficients of $(KX)=0$.† An invariant condition obtained for one form of the equations of the curve may readily be expressed in the invariants of the other form. The relations connecting the two systems are

$$(1)\qquad \begin{cases} 4I_2 = MS_1S_3, \\ I_2' = M(S_1S_3 - 16S_4), \\ I_4 = (16M)^2(S_2S_3^2 - S_1S_3S_4 + S_4^2), \\ I_6 = M^3S_3^2(S_1S_2S_3 - S_1^2S_4 - S_3^2), \end{cases}$$

where M is a positive constant. These equations may also be solved for any function of S_1, S_2, S_3, S_4, whose weight is a multiple of four.‡

To determine the nature of the nodes from the invariants, it is desirable to refer the quartic to a triangle whose vertices are at the nodes. Such a quartic is given by the equations§

$$X_0^2X_1^2 + X_1^2X_2^2 + X_0^2X_2^2 - 2X_0X_1X_2X_3 = 0,$$

$$\alpha_0X_0 + \alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3 = 0.$$

Eliminating X_3 we have

$$(2)\qquad \alpha_3(X_0^2X_1^2 + X_1^2X_2^2 + X_0^2X_2^2) + 2X_0X_1X_2(\alpha_0X_0 + \alpha_1X_1 + \alpha_2X_2) = 0,$$

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† J. E. Rowe, Transactions of this Society, vol. 12 (1911), pp. 295–310.

‡ L. T. Moore, American Journal, vol. 48 (1926), p. 251.

§ The surface $(\sqrt{X})=0$ referred to a tetrahedron having the three double lines as edges. Salmon-Rogers, *Geometry of Three Dimensions*, vol. 2, p. 213.

which may be derived from the conic

$$(3)\quad \alpha_3(X_0^2 + X_1^2 + X_2^2) + 2\alpha_2 X_0X_1 + 2\alpha_1 X_0X_2 + 2\alpha_0 X_1X_2 = 0,$$

by the transformation $X_0 = 1/X_0'$; $X_1 = 1/X_1'$; $X_2 = 1/X_2'$.

The points of intersection of (3) with the sides of the triangle of reference correspond to the nodes of (2). Therefore, if (3) cuts a side in two real, coincident, or imaginary points, the corresponding node is, respectively, a crunode, a cusp, or an acnode.*

It is evident that the expressions $(\alpha_0^2 - \alpha_3^2)$, $(\alpha_1^2 - \alpha_3^2)$, $(\alpha_2^2 - \alpha_3^2)$ determine the nature of the nodes, and from a consideration of the functions

$$F_3(\alpha) = (\alpha_0^2 - \alpha_3^2)(\alpha_1^2 - \alpha_3^2)(\alpha_2^2 - \alpha_3^2),$$
$$F_2(\alpha) = (\alpha_0^2 - \alpha_3^2)(\alpha_1^2 - \alpha_3^2) + (\alpha_0^2 - \alpha_3^2)(\alpha_2^2 - \alpha_3^2) + (\alpha_1^2 - \alpha_3^2)(\alpha_2^2 - \alpha_3^2),$$
$$F_1(\alpha) = (\alpha_0^2 - \alpha_3^2) + (\alpha_1^2 - \alpha_3^2) + (\alpha_2^2 - \alpha_3^2),$$

the following classification can be obtained:

Three crunodes: $F_3(\alpha) > 0$, $F_2(\alpha) > 0$, $F_1(\alpha) > 0$;

Two crunodes, one acnode: $F_3(\alpha) < 0$, $\{F_1(\alpha) > 0$ or $F_1(\alpha) < 0$, $F_2(\alpha) < 0\}$;

Three acnodes: $F_3(\alpha) < 0$, $F_2(\alpha) > 0$, $F_1(\alpha) < 0$;

One crunode, two acnodes: $F_3(\alpha) > 0$, $F_2(\alpha)$ or $F_1(\alpha) < 0$.

The following relations connect the coefficients of $(\alpha X) = 0$ and $(KX) = 0$:†

$$S_1 = K_0 + K_1 + K_2 + K_3 = 4\alpha_3,$$
$$S_2 = K_0K_1 + K_0K_2 + K_0K_3 + K_1K_2 + K_1K_3 + K_2K_3 = 2(3\alpha_3^2 - \alpha_0^2 - \alpha_1^2 - \alpha_2^2),$$
$$S_3 = 4(\alpha_3^3 - \alpha_3(\alpha_0^2 + \alpha_1^2 + \alpha_2^2) + 2\alpha_0\alpha_1\alpha_2),$$
$$S_4 = \alpha_0^4 + \alpha_1^4 + \alpha_2^4 + \alpha_3^4 - 2\alpha_1^2\alpha_0^2 - 2\alpha_0^2\alpha_2^2 - 2\alpha_0^2\alpha_3^2 - 2\alpha_1^2\alpha_2^2 - 2\alpha_1^2\alpha_3^2 - 2\alpha_2^2\alpha_3^2 + 8\alpha_0\alpha_1\alpha_2\alpha_3.$$

* Salmon, *Higher Plane Curves*, 1879, p. 254.

† L. T. Moore, loc. cit., p. 248.

From these relations we have

$$64F_3(\alpha) = (S_1^2S_4 - S_1S_2S_3 + S_3^2),\ 16F_2(\alpha) = (S_2^2 + S_1S_3 - 4S_4),$$
$$2F_1(\alpha) = -S_2.$$

Again, after multiplying $F_1(\alpha)$ by S_1^2 and $F_3(\alpha)$ by S_3^2, we have from equations (1),

$$M^3S_3^2(S_1^2S_4 - S_1S_2S_3 + S_3^2) = -I_6,$$

$$1024M(S_2^2 + S_1S_3 - 4S_4)$$
$$= \frac{\{I_4 + 64I_2(4I_2 - I_2') - (4I_2 - I_2')^2\}^2 + 256I_6'(I_2' + 12I_2)}{I_6'},$$

$$MS_1^2S_2 = \frac{4I_2^2\{I_4 + 64I_2(4I_2 - I_2') - (4I_2 - I_2')^2}{I_6'},$$

where $I_6' = I_2I_4 - I_2(I_2' - 4I_2)^2 - 64I_6$.

Thus we have established sufficient criteria, in each of the three systems of invariants, to determine the nature of the nodes of the curve.

We can determine the nature of the nodes of rational tacnodal and oscnodal quartics by the same method used for trinodal quartics.* The results obtained may be stated as follows:

Tacnode and crunode: $I_6 < 0,\ I_2' - 4I_2 < 0$;
Tacnode and acnode: $I_6 > 0,\ I_2' - 4I_2 < 0$;
Isolated tacnode and crunode: $I_6 < 0,\ I_2' - 4I_2 > 0$;
Isolated tacnode and acnode: $I_6 > 0,\ I_2' - 4I_2 > 0$;
Isolated oscnode: $I_6 < 0$.

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* L. T. Moore and J. H. Neelley, *Rational tacnodal and oscnodal quartic curves considered as plane sections of quartic surfaces*, to appear soon in the American Journal. In this paper, equations corresponding to equations (1), (2), (3) above are developed for tacnodal and oscnodal quartics.